

Strict monotonicity of principal eigenvalues of elliptic operators in \mathbb{R}^d and risk-sensitive control

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Abstract This paper studies the eigenvalue problem on \mathbb{R}^d for a class of second order, elliptic operators of the form $\mathcal{L}^f = a^{ij} \partial_{x_i} \partial_{x_j} + b^i \partial_{x_i} + f$, associated with non-degenerate diffusions. We present various results that relate the strict monotonicity of the principal eigenvalue of the operator with respect to the potential function f to the ergodic properties of the corresponding ‘twisted’ diffusion, and provide sufficient conditions for this monotonicity property to hold. Based on these characterizations, we extend or strengthen various results in the literature for a class of viscous Hamilton–Jacobi–Bellman equations of ergodic type to equations with measurable drift and potential. In addition, we establish the strong duality for the equivalent infinite dimensional linear programming formulation of these ergodic control problems. We also apply these results to the study of the infinite horizon risk-sensitive control problem for diffusions. We establish existence of optimal Markov controls, verification of optimality results, and the continuity of the controlled principal eigenvalue with respect to stationary Markov controls.

Keywords generalized principal eigenvalue · recurrence and transience · viscous Hamilton–Jacobi equation · risk-sensitive control · ergodic control · nonlinear eigenvalue problems

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1 Introduction

In this paper we study the eigenvalue problem on \mathbb{R}^d for non-degenerate, second order elliptic operators \mathcal{L}^f of the form

$$\mathcal{L}^f \varphi = \sum_{i,j=1}^d a^{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^d b^i \frac{\partial \varphi}{\partial x_i} + f \varphi. \quad (1.1)$$

Here $b, f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, $a \in C^{0,1}_{\text{loc}}(\mathbb{R}^d)$ and a, b satisfy a linear growth assumption in the outward radial direction (see (A2) in [subsection 1.1](#)). In other words, a and b satisfy the usual assumptions for existence and uniqueness of a strong solution of the Itô equation

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s, \quad \text{with} \quad a = \frac{1}{2} \sigma \sigma^\top, \quad (1.2)$$

where W is a standard Brownian motion.

We focus on certain properties of the principal eigenvalue of the operator \mathcal{L}^f which play a key role in infinite horizon risk-sensitive control problems. When D is a smooth bounded domain and a, b, f are regular enough, existence of a principal eigenvalue, and corresponding eigenfunction under a Dirichlet boundary condition can be obtained by an application of Krein-Rutman theory (see for instance [\[31, 38\]](#)). This eigenvalue is the bottom of the spectrum of $-\mathcal{L}^f$ with Dirichlet boundary condition. When D is non smooth, a generalized notion of a principal eigenvalue was introduced in the seminal work of Berestycki, Nirenberg and Varadhan [\[6\]](#). An analogous theory for non-linear elliptic operators has been developed by Quaas and Sirakov in [\[39\]](#). The principal eigenvalue plays a key role in the study of non-homogeneous elliptic operators and the maximum principle (see [\[6, 8, 21, 39\]](#)). For some other definitions of the principal (or *critical*) eigenvalue we refer the reader to the works of Pinchover [\[37\]](#) and Pinsky [\[38, Chapter 4\]](#).

For unbounded domains D , principal eigenvalue problems have been recently considered by Berestycki and Rossi in [\[7, 8\]](#). Not surprisingly, certain properties of the principal eigenvalue which hold in bounded domains may not be true for unbounded ones. For instance, when D is smooth and bounded it is well known that for the Dirichlet boundary value problem, the principal eigenvalue is simple, and the associated principal eigenfunction is positive. Moreover, it is the unique eigenvalue with a positive eigenfunction. But if D is unbounded and smooth, then there exists a constant $\lambda^* = \lambda^*(f)$ such that any $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of \mathcal{L}^f with a positive eigenfunction [\[8, Theorem 1.4\]](#) (see also [\[21\]](#) and [\[30, Theorem 2.6\]](#)). The lowest such value λ^* serves as a definition of the principal eigenvalue when D is not bounded. The principal eigenvalue is known to be strictly monotone as a function of the bounded domain D (the latter ordered with respect to set inclusion), and also strictly monotone in the coefficient f when D is bounded

(see [8] and Lemma 2.1 below). These properties fail to hold in unbounded domains as remarked by Berestycki and Rossi [8, Remark 2.4]. Strict monotonicity of $f \mapsto \lambda^*(f)$ and its implications, is a central theme in our study. We adopt a probabilistic approach in our investigation. One can view $\lambda^*(f)$ as a risk-sensitive average of f over the diffusion in (1.2). More precisely, since (1.2) has a unique solution which exists for all $t \in [0, \infty)$, then we can define

$$\mathcal{E}_x(f) := \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x \left[e^{\int_0^T f(X_s) ds} \right], \quad x \in \mathbb{R}^d, \quad (1.3)$$

with ‘log’ denoting the natural logarithm. As shown in the proof of Lemma 2.3 in [1] we have $\lambda^*(f) \leq \mathcal{E}_x(f)$, and equality is indeed the case in many important situations, although strictly speaking it is only a heuristic. This heuristic is based on the fact that for a bounded f , the operator \mathcal{L}^f is the infinitesimal generator of a strongly continuous, positive semigroup with potential f , see for instance [19, Chapter IV]. Now if f is a bounded continuous function and if the occupation measures of $\{X_t\}$ obey a large deviation principle, then one can express $\mathcal{E}_x(f)$ in terms of the large deviation rate function. This is known as the variational representation for the eigenvalue. See for instance the article by Donsker and Varadhan [17] where this representation is obtained for compact domains. But large deviation principles for $\{X_t\}$ are generally available only under strong hypotheses on the process (see [18]). In this paper we rely on the stochastic representation of the principal eigenfunction which can be established under very mild hypotheses. This approach is recently used by Arapostathis and Biswas in [1] to study the multiplicative Poisson equation when f is *near-monotone* (which includes the case of inf-compact f). By an *eigenpair* of \mathcal{L}^f we mean a pair (Ψ, λ) , with Ψ a positive function in $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, for all $p \in [1, \infty)$, and $\lambda \in \mathbb{R}$, that satisfies

$$\mathcal{L}^f \Psi = a^{ij} \partial_{ij} \Psi + b^i \partial_i \Psi + f \Psi = \lambda \Psi. \quad (1.4)$$

We refer to λ as the *eigenvalue* and to Ψ as the *eigenfunction*. In (1.4) we adopt the notation $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ for $i, j \in \mathbb{N}$, and use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d .

As mentioned earlier, such a pair (Ψ, λ) exists only if $\lambda \geq \lambda^*(f)$ (see Corollary 2.1). Given an eigenpair (Ψ, λ) , the associated *twisted* diffusion Y is an Itô process as in (1.2), but with the drift b replaced by $b + 2a\nabla(\log \Psi)$. It is not generally the case that the twisted process has a strong solution which exists for all time. If $\lambda > \lambda^*(f)$ the twisted diffusion is always transient (see Lemma 2.6). When $\lambda = \lambda^*(f)$, the eigenfunction is denoted as Ψ^* and is called the *ground state* [38, 41]. The corresponding twisted diffusion, denoted by Y^* , is referred to as the *ground-state diffusion*.

Let $C_0^+(\mathbb{R}^d)$ ($C_c^+(\mathbb{R}^d)$) denote the class of non-zero, nonnegative real valued continuous functions on \mathbb{R}^d which vanish at infinity (have compact support). We say that $\lambda^*(f)$ is *strictly monotone at f* if there exists $h \in C_0^+(\mathbb{R}^d)$ satisfying $\lambda^*(f - h) < \lambda^*(f)$. We also say that $\lambda^*(f)$ is *strictly monotone at f on the right* if $\lambda^*(f + h) > \lambda^*(f)$ for all $h \in C_c^+(\mathbb{R}^d)$. In Theorem 2.1 we show that strict monotonicity at f implies strict monotonicity at f on the right. Our main results provide sharp characterizations of the ground state Ψ^* and the ground state process Y^* in terms of these monotonicity properties. Assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a locally bounded, Borel measurable function, satisfying $\text{ess inf}_{\mathbb{R}^d} f > -\infty$, and that $\lambda^*(f)$ is finite. We show that strict monotonicity of $\lambda^*(f)$ at f on the right implies the simplicity of $\lambda^*(f)$, i.e., the uniqueness of the ground state Ψ^* , and that this is also a necessary and sufficient condition for the ground state process to be recurrent (see Lemma 2.7 and Theorem 2.3). Another important result is that the ground state diffusion is exponentially ergodic (see Definition 2.2) if and only if $\lambda^*(f)$ is strictly monotone at f . These results are summarized in Theorem 2.1 in section 2. Other results in section 2 provide a characterization of the eigenvalue in terms of the long time behavior of the twisted process and stochastic representations of the ground state (see Lemmas 2.2, 2.3 and 2.7 and Theorem 2.4).

In [38], Pinsky uses the existence of a Green’s measure to define the *critical* eigenvalue of a non-degenerate elliptic operator. This critical eigenvalue coincides with the principal eigenvalue when the boundary of the domain and the coefficients of \mathcal{L}^f are smooth enough. It is shown that for any bounded domain, and provided that the coefficients are in $C^{1,\alpha}(\mathbb{R}^d)$, $\alpha > 0$, and bounded, there exists a critical value λ_c such that for any $\lambda > \lambda_c$ we can find a Green’s

measure corresponding to the operator $\mathcal{L}^{(f-\lambda)}$ [38, Theorem 4.7.1]. The result in [Theorem 2.3](#) in [section 2](#) extends this to \mathbb{R}^d without assuming much regularity on the coefficients. Let us also remark that most of our results in [section 2](#) hold for general unbounded domains with smooth boundary.

Continuous dependence of λ^* on the coefficients of \mathcal{L} has also been a topic of interest. It is not hard to see that $f \mapsto \lambda^*(f)$ is lower-semicontinuous in the $L^1_{\text{loc}}(\mathbb{R}^d)$ topology for f . Continuity of this map is also established in [8, Proposition 9.2] with respect to the $L^\infty(\mathbb{R}^d)$ norm on f . In [Theorem 2.4](#) and [Remark 4.1](#) we study the continuity property of $\lambda^*(f)$ for a class of f but under $L^1_{\text{loc}}(\mathbb{R}^d)$ topology. We also obtain a pinned multiplicative ergodic theorem which is of independent interest, and show that $\mathcal{E}_x(f) = \lambda^*(f)$ for a large class of problems.

We next discuss the connection of this problem with a stochastic ergodic control problem. Defining $\check{\Psi} = \log \Psi^*$ we obtain from (1.4) that

$$\begin{aligned} a^{ij} \partial_{ij} \check{\Psi} + b^i \partial_i \check{\Psi} - \langle \nabla \check{\Psi}, a \nabla \check{\Psi} \rangle &= a^{ij} \partial_{ij} \check{\Psi} + b^i \partial_i \check{\Psi} + \min_{u \in \mathbb{R}^d} [2 \langle a u, \nabla \check{\Psi} \rangle + \langle u, a u \rangle] \\ &= f - \lambda^*(f). \end{aligned} \tag{1.5}$$

It is easy to see that (1.5) is related to an ergodic control problem with controlled drift $(b + 2a \cdot u)$ and running cost $\langle u, a u \rangle - f(x)$. The parameter $\lambda^*(f)$ can be thought of as the optimal ergodic value; see Ichihara [27]. It should be observed that the twisted process defined above corresponds to the optimally controlled diffusion. We refer to Ichihara [26, 27] and Kaise and Sheu [30] for some existing works in this direction. For a potential f that vanishes at infinity, Ichihara [27, 28] considers the ergodic control problem in (1.5), with a more general Hamiltonian and under scaling of the potential. When f is nonnegative, it is shown that the value of the ergodic problem with potential βf , $\beta \in \mathbb{R}$, equals the eigenvalue $\lambda^*(\beta f)$ when the parameter β exceeds a critical value β_c and $\nabla \Psi^*$ is the optimal control, and below that critical value a bifurcation occurs. Analogous are the results in [4] for viscous Hamilton–Jacobi equation with a the identity matrix and a Hamiltonian which is a power of the gradient term. These results are obtained for bounded, and Lipschitz continuous a , b , and f . In [Theorems 2.5](#) and [2.6](#) we extend these results to measurable b and f , and possibly unbounded a and b .

Optimality for the ergodic problem is shown in [27, 28] via the study of the optimal finite horizon problem (Cauchy parabolic problem). Inevitably, in doing so, optimality is shown in a certain class of controls. To overcome this limitation, we take a different approach to the ergodic control problem in (1.5). As well known, ergodic control problems can be cast as infinite dimensional linear programs [9, 40]. Consider a controlled diffusion, with the control taking values in as space \mathbb{U} with extended generator \mathcal{A} , where the ‘action’ $u \in \mathbb{U}$ enters implicitly as a parameter in \mathcal{A} . Let $\mathcal{R}: \mathbb{U} \rightarrow \mathbb{R}$ denote the running cost. The primal problem then can be written

$$\alpha_* = \left\{ \inf \int_{\mathbb{R}^d \times \mathbb{U}} \mathcal{R}(x, u) \pi(dx, du) : \mathcal{A}^* \pi = 0, \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U}) \right\}.$$

Here $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ denotes the class of probability measures on the Borel σ -field of $\mathbb{R}^d \times \mathbb{U}$, and π is an *ergodic occupation measure* (see [9]). The dual problem takes the form

$$\alpha = \sup \left\{ c \in \mathbb{R} : \inf_{u \in \mathbb{U}} [\mathcal{A} g(x, u) + \mathcal{R}(x, u)] \geq c, g \in \mathcal{D}(\mathcal{A}) \right\},$$

where $\mathcal{D}(\mathcal{A})$ denotes the domain of \mathcal{A} . In other words the dual problem is a maximization over subsolutions of the Hamilton–Jacobi–Bellman (HJB) equation. For non-degenerate diffusions with a compact action space \mathbb{U} , under the hypothesis that \mathcal{R} is near-monotone, or under uniform ergodicity conditions, it is well known that we have strong duality, i.e., $\alpha_* = \alpha$. To the best of our knowledge, this not been established for problems with non-compact action spaces. In [Theorem 2.7](#) we establish strong duality for the ergodic problem in (1.5). In this result, the coefficients b and f are bounded and measurable, and a is bounded, Lipschitz, and uniformly elliptic. Moreover, we establish the unicity of the optimal ergodic occupation measure, and as a result of this, the uniqueness of the optimal stationary Markov

control. The methodology is general enough that can be applied to various classes of ergodic control problems that are characterized by viscous HJB equations.

The results in [26, 30] are obtained for regular coefficients ($C^{2,\alpha}$ type), and under an assumption of exponential ergodicity (see (3.12) below). We provide a sufficient condition in (H2) under which strict monotonicity of the principal eigenvalue holds. It is also shown that the exponential ergodicity condition of [26, 30] actually implies (H2); thus (H2) is weaker. Moreover, exponential stability (of (3.12) type) does not hold for bounded coefficients a and b . See Remark 3.4 for details. In Theorem 3.3 we cite a sufficient condition under which strict monotonicity of $\lambda^*(f)$ holds even when a and b are bounded. Let us also remark that the regularity assumption in [26, 30] on the coefficients a , b and f can not be waived if one follows their method. This is because the work in [26, 30] relies on a gradient estimate (Bernstein method) which is not available under weaker regularity settings. But this amount of regularity might not be available in many situation, for instance in models with a measurable drift which are often encountered in stochastic control problems. Let us also mention the unpublished work of Kaise and Sheu in [29] that contains some results similar to ours, in particular, similar to the results in section 3 and pinned multiplicative ergodic theorems. The results in [29] are obtained under stronger regularity assumptions on the coefficients a , b and f .

In section 4 we apply the above mentioned results to study the infinite horizon risk-sensitive control problem. We refer the reader to [1] where the importance of these control problems is discussed. Unfortunately the development of the infinite horizon risk-sensitive control problem for controlled diffusions has not been completely satisfactory, and the same applies to controlled Markov chain models. Most of the available results have been obtained under restrictive settings, and a full characterization including uniqueness of the solution to the risk-sensitive HJB equation and verification of optimality results is lacking. Let us give a quick overview of the existing literature in the settings of controlled diffusions and relevant to our problem. Risk-sensitive control for models with a constant diffusion matrix and *asymptotically flat* type drift is studied by Fleming and McEneaney in [20]. Another particular setting is considered by Nagai [36], where the action space is the whole Euclidean space and the running cost has a specific structure. Menaldi and Robin have considered models with periodic data [35]. Under the assumption of a near-monotone cost, the infinite horizon risk-sensitive control problem is studied in [1, 10, 12], whereas Biswas in [11] has considered this problem under the assumption of exponential ergodicity. Differential games with risk-sensitive type costs have been studied by Basu and Ghosh [5], and Ghosh et. al. [22]. All the above studies, have obtained existence of a pair (V, λ^*) that satisfies the risk-sensitive HJB equation, with λ^* the optimal risk-sensitive value, and show that any minimizing selector of the HJB is an optimal control. The works in [35, 36] address the existence and uniqueness of a solution to the HJB equation, in their particular set up, but do not contain any verification of optimality results. Two main results that are missing from the existing literature, with the exception of [1], are (a) uniqueness of value function, and (b) verification for optimal control.

Following the ergodic control paradigm, we can identify two classes of models: (i) models with a near-monotone running cost and finite optimal value, and no other hypotheses on the dynamics, and (ii) models that enjoy a uniform exponential ergodicity. Near-monotone running cost models are studied in [1, 10, 12, 36]; however, only [1] obtains a full characterization without imposing a blanket ergodicity hypothesis. Studies for models in class (ii) can be found in [5, 11, 20, 22].

In this paper we study models in class (ii). The results developed in sections 2 and 3 enable us to obtain a full characterization of the risk-sensitive control problem in section 4. The main hypotheses are in Assumptions 4.1 and 4.2. Another interesting result that we establish in section 4 is the continuity of the controlled principal eigenvalue with respect to (relaxed) stationary Markov controls (see Theorem 4.3). This facilitates establishing the existence of an optimal stationary Markov control for risk-sensitive control problems under risk-sensitive type constraints. This is done in Theorem 4.3. Let us also remark that this existence result was not known, since the controlled risk-sensitive value is lower-semicontinuous with respect to Markov controls and the equality $\lambda^*(f) = \mathcal{E}(f)$ is not true in general.

Moreover, the usual technique of Lagrange multipliers does not work in this situation, because of the non-convex nature of the optimization criterion.

To summarize the main contributions of the paper, we have

- (a) established several characterizations of the property of strict monotonicity of the principal eigenvalue,
- (b) extended several results in the literature on viscous HJB equations with potentials f vanishing at infinity to measurable potentials and measurable drift,
- (c) studied a general class of risk-sensitive control problems under a uniform ergodicity hypothesis, and established uniqueness of a solution to the HJB equation and verification of optimality results,
- (d) established continuity results of the controlled principal eigenvalue with respect to stationary Markov controls.

The paper is organized as follows. [Subsection 1.1](#) states the assumptions on the coefficients of \mathcal{L} , and [subsection 1.2](#) summarizes the notation used in the paper. [Subsection 1.3](#) summarizes some basic results from the theory of elliptic pdes. [Section 2](#) contains results on the principal eigenvalue under minimal assumptions, in its first three subsections, and [subsection 2.4](#) is devoted to operators with potential f which vanishes at infinity. [Section 3](#) improves on the results of [section 2](#), under the assumption that [\(1.2\)](#) is exponentially ergodic. [Section 4](#) is dedicated to the infinite horizon, risk-sensitive optimal control problem.

1.1 Assumptions on the model

The following assumptions on the coefficients of \mathcal{L} are in effect throughout the paper unless otherwise mentioned.

- (A1) *Local Lipschitz continuity*: The function $\sigma = [\sigma^{ij}] : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, with $\|\sigma\| := \sqrt{\text{trace}(\sigma\sigma^\top)}$, we have

$$\|\sigma(x) - \sigma(y)\| \leq C_R |x - y| \quad \forall x, y \in B_R.$$

We also assume that $b = [b^1, \dots, b^d]^\top : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally bounded and measurable.

- (A2) *Affine growth condition*: b and σ satisfy a global growth condition of the form

$$\langle b(x), x \rangle^+ + \|\sigma(x)\|^2 \leq C_0(1 + |x|^2) \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_0 > 0$.

- (A3) *Nondegeneracy*: For each $R > 0$, it holds that

$$\sum_{i,j=1}^d a^{ij}(x) \xi_i \xi_j \geq C_R^{-1} |\xi|^2 \quad \forall x \in B_R,$$

and for all $\xi = (\xi_1, \dots, \xi_d)^\top \in \mathbb{R}^d$, where, as defined earlier, $a := \frac{1}{2} \sigma \sigma^\top$.

Let us remark that the assumptions (A1)–(A3) are not optimal, and can be weakened in many situations. For instance, if σ is continuous and its weak derivative lies in $L_{\text{loc}}^{2(d+1)}(\mathbb{R}^d)$, then [\(2.1\)](#) has a unique strong solution (see [\[43\]](#)). All of our results below can be extended to this setup as well.

1.2 Notation

The standard Euclidean norm in \mathbb{R}^d is denoted by $|\cdot|$, and $\langle \cdot, \cdot \rangle$ denotes the inner product. The set of nonnegative real numbers is denoted by \mathbb{R}_+ , \mathbb{N} stands for the set of natural numbers, and $\mathbf{1}$ denotes the indicator function. Given two real numbers a and b , the minimum (maximum) is denoted by $a \wedge b$ ($a \vee b$), respectively. The closure, boundary, and the complement of a set $A \subset \mathbb{R}^d$ are denoted by \bar{A} , ∂A , and A^c , respectively. We denote by $\tau(A)$ the *first exit time* of the process $\{X_t\}$ from the set $A \subset \mathbb{R}^d$, defined by

$$\tau(A) := \inf \{t > 0 : X_t \notin A\}.$$

The open ball of radius r in \mathbb{R}^d , centered at the origin, is denoted by B_r , and we let $\tau_r := \tau(B_r)$, and $\check{\tau}_r := \tau(B_r^c)$.

The term *domain* in \mathbb{R}^d refers to a nonempty, connected open subset of the Euclidean space \mathbb{R}^d . For a domain $D \subset \mathbb{R}^d$, the space $C^k(D)$ ($C^\infty(D)$), $k \geq 0$, refers to the class of all real-valued functions on D whose partial derivatives up to order k (of any order) exist and are continuous. Also, $C_b^k(D)$ ($C_b^\infty(D)$) is the set of functions whose partial derivatives up to order k (of any order) are continuous and bounded in D , and $C_c^k(D)$ the class of functions in $C^k(D)$, $0 \leq k \leq \infty$, that have compact support. In addition, $C_0(\mathbb{R}^d)$ denotes the class of continuous functions on \mathbb{R}^d that vanish at infinity. By $C_c^+(\mathbb{R}^d)$ and $C_0^+(\mathbb{R}^d)$ we denote the subsets of $C_c(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$, respectively, consisting of all non-trivial nonnegative functions. We use the term non-trivial to refer to a function that is not a.e. equal to 0. The space $L^p(D)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes of) measurable functions f satisfying $\int_D |f(x)|^p dx < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in D . The standard Sobolev space of functions on D whose generalized derivatives up to order k are in $L^p(D)$, equipped with its natural norm, is denoted by $\mathcal{W}^{k,p}(D)$, $k \geq 0$, $p \geq 1$. For a probability measure μ in $\mathcal{P}(\mathbb{R}^d)$ and a real-valued function f which is integrable with respect to μ we use the notation

$$\langle f, \mu \rangle = \mu(f) := \int_{\mathbb{R}^d} f(x) \mu(dx).$$

In general, if \mathcal{X} is a space of real-valued functions on Q , \mathcal{X}_{loc} consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in C_c^\infty(Q)$, the space of smooth functions on Q with compact support. In this manner we obtain for example the space $\mathcal{W}_{\text{loc}}^{2,p}(Q)$.

We adopt the notation $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ for $i, j \in \{1, \dots, d\}$, and we often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d .

Throughout the paper we use basic results for elliptic pdes which we describe next.

1.3 Some basic results from the theory of second order elliptic pdes

Harnack's inequality plays a central role in the study of elliptic equations, and can be stated as follows [23, Theorem 9.1]. Suppose that $\phi \in \mathcal{W}_{\text{loc}}^{2,p}(B_{R+1})$, $p > d$, $R > 0$, is a positive function that solves $\mathcal{L}\phi + h\phi = 0$ on B_{R+1} , with $h \in L^\infty(B_{R+1})$. Then there exists a constant C_H depending only on R , d , the constants C_{R+1} and C_0 in (A1)–(A3), and $\|h\|_\infty$ such that

$$\phi(x) \leq C_H \phi(y) \quad \forall x, y \in B_R.$$

Relative weak compactness of a family of functions in $\mathcal{W}_{\text{loc}}^{2,p}(B_{R+1})$ can be obtained as a result of the following well-known a priori estimate [16, Lemma 5.3]. If $\varphi \in \mathcal{W}_{\text{loc}}^{2,p}(B_{R+1}) \cap L^p(B_{R+1})$, with $p \in (1, \infty)$, then

$$\|\varphi\|_{\mathcal{W}^{2,p}(B_R)} \leq C \left(\|\varphi\|_{L^p(B_{R+1})} + \|\mathcal{L}\varphi\|_{L^p(B_{R+1})} \right)$$

with the constant C depending only on d , R , C_{R+1} , and C_0 . This estimate along with the compactness of the embedding $\mathcal{W}^{2,d}(B_R) \hookrightarrow C^{1,r}(\bar{B}_R)$, for $p > d$ and $r < 1 - \frac{d}{p}$ (see [16, Proposition 1.6]) imply the equicontinuity of any family of

functions φ_n which satisfies $\sup_n (\|\varphi_n\|_{L^p(B_{R+1})} + \|\mathcal{L}\varphi_n\|_{L^p(B_{R+1})}) < \infty$, $p > d$. We also frequently use the weak and strong maximum principles in the following form [23, Theorems 9.5 and 9.6, p. 225]. The *weak maximum principle* states that if $\varphi, \psi \in \mathcal{W}_{\text{loc}}^{2,d}(D) \cap C(\bar{D})$ satisfy $(\mathcal{L} + h)\varphi = (\mathcal{L} + h)\psi$ in a bounded domain $D \subset \mathbb{R}^d$, with $h \in L^d(D)$, $h \leq 0$ and $\varphi = \psi$ on ∂D , then $\varphi = \psi$ in D . On the other hand, the *strong maximum principle* states that if $\varphi \in \mathcal{W}_{\text{loc}}^{2,d}(D)$ satisfies $(\mathcal{L} + h)\varphi \leq 0$ in a bounded domain D , with $h = 0$ ($h \leq 0$), then φ cannot attain a minimum (non-positive minimum) in D unless it is a constant. We often use the following variation of the strong maximum principle. If $\varphi \geq 0$ and $(\mathcal{L} + h)\varphi \leq 0$ in a bounded domain $D \subset \mathbb{R}^d$, then φ is either positive on D or identically equal to 0. This follows from the general statement by writing $(\mathcal{L} + h)\varphi \leq 0$ as

$$(\mathcal{L} - h^-)\varphi \leq -h^+\varphi \leq 0.$$

In addition, we often use Krylov's extension of Itô's formula for functions in $\mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ which we refer to as the Itô–Krylov formula [32, p. 122].

2 General Results

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, \mathbb{P})$ be a given filtered probability space with a complete, right continuous filtration $\{\mathfrak{F}_t\}$. Let W be a standard Brownian motion adapted to $\{\mathfrak{F}_t\}$. Consider the stochastic differential equation

$$X_t = X_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s. \quad (2.1)$$

The third term on the right hand side of (2.1) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (2.1), if it is \mathfrak{F}_t -adapted, continuous in t , defined for all $\omega \in \Omega$ and $t \in [0, \infty)$, and satisfies (2.1) for all $t \in [0, \infty)$ a.s. It is well known that under (A1)–(A3), there exists a unique solution of (2.1) [2, Theorem 2.2.4]. We let \mathbb{E}_x denote the expectation operator on the canonical space of the process with $X_0 = x$, and \mathbb{P}_x the corresponding probability measure. Recall that $\tau(D)$ denotes the first exit time of the process X from a domain D . The process X is said to be *recurrent* if for any bounded domain D we have $\mathbb{P}_x(\tau(D^c) < \infty) = 1$ for all $x \in \bar{D}^c$. Otherwise the process is called transient. A recurrent process is said to be *positive recurrent* if $\mathbb{E}_x[\tau(D^c)] < \infty$ for all $x \in \bar{D}^c$. It is known that for a non-degenerate diffusion the property of recurrence (or positive recurrence) is independent of D and x , i.e., if it holds for some domain D and $x \in \bar{D}^c$, then it also holds for every domain D , and all points $x \in \bar{D}^c$ (see [2, Lemma 2.6.12 and Theorem 2.6.10]). We define the extended operator $\mathcal{L} : C^2(\mathbb{R}^d) \mapsto L_{\text{loc}}^\infty(\mathbb{R}^d)$ associated to (2.1) by

$$\mathcal{L}g(x) = a^{ij}(x) \partial_{ij}g(x) + b^i(x) \partial_i g(x). \quad (2.2)$$

We let $\mathcal{L}^f := \mathcal{L} + f$, for a function f which is called *the potential*, and satisfies the following:

(H1) $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a locally bounded, Borel measurable function, satisfying $\inf_{\mathbb{R}^d} f > -\infty$, and $\lambda^*(f)$ is finite.

Hypothesis (H1) is enforced throughout section 2 without further mention, and it is repeated only for emphasis.

2.1 Risk-sensitive value and Dirichlet eigenvalues

For a potential f satisfying (H1) we define

$$\mathcal{E}_x(f) := \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x \left[e^{\int_0^T f(X_s) ds} \right], \quad \text{and} \quad \mathcal{E}(f) := \inf_{x \in \mathbb{R}^d} \mathcal{E}_x(f). \quad (2.3)$$

We refer to $\mathcal{E}(f)$ as the *risk-sensitive average* of f . This quantity plays a key role in our analysis.

The following lemma summarizes some results from [6, 8, 39] on the eigenvalue of the Dirichlet problem for the operator \mathcal{L}^f . For simplicity, we state it for balls B_r , instead of more general domains.

Lemma 2.1 *For each $r \in (0, \infty)$ there exists a unique pair $(\widehat{\Psi}_r, \hat{\lambda}_r) \in (\mathcal{W}_{\text{loc}}^{2,p}(B_r) \cap C(\bar{B}_r)) \times \mathbb{R}$, for any $p \in [1, \infty)$, satisfying $\widehat{\Psi}_r > 0$ on B_r , $\widehat{\Psi}_r = 0$ on ∂B_r , and $\widehat{\Psi}_r(0) = 1$, which solves*

$$\mathcal{L}\widehat{\Psi}_r(x) + f(x)\widehat{\Psi}_r(x) = \hat{\lambda}_r\widehat{\Psi}_r(x) \quad \text{a.e. } x \in B_r, \quad (2.4)$$

with \mathcal{L} as defined in (2.2). Moreover, $\hat{\lambda}_r$ has the following properties:

- (a) *The map $r \mapsto \hat{\lambda}_r$ is continuous and strictly increasing.*
- (b) *In its dependence on the function f , $\hat{\lambda}_r$ is nondecreasing, convex, and Lipschitz continuous (with respect to the L^∞ norm) with Lipschitz constant 1. In addition, if $f \not\geq f'$ then $\hat{\lambda}_r(f) < \hat{\lambda}_r(f')$.*

Proof Existence and uniqueness of the solution follow by [39, Theorem 1.1] (see also [6]). Part (a) follows by [8, Theorem 1.10], and (iii)–(iv) of [8, Proposition 2.3] while part (b) follows by [6, Proposition 2.1]. \square

We refer to $(\widehat{\Psi}_r, \hat{\lambda}_r)$ as the *eigensolution* of the Dirichlet problem, or *Dirichlet eigensolution* of \mathcal{L}^f on B_r . Correspondingly, $\hat{\lambda}_r$ and $\widehat{\Psi}_r$ are referred to as the *Dirichlet eigenvalue* and *Dirichlet eigenfunction*, respectively.

Lemma 2.1 (a) motivates the following definition.

Definition 2.1 The principal eigenvalue $\lambda^*(f)$ on \mathbb{R}^d of the operator \mathcal{L}^f given in (1.1) is defined as $\lambda^*(f) := \lim_{r \rightarrow \infty} \hat{\lambda}_r(f)$.

We compare Definition 2.1 with the following definition for the principal eigenvalue, commonly used in the pde literature [8].

$$\hat{\Lambda}(f) = \inf \left\{ \lambda \in \mathbb{R} : \exists \varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d), \varphi > 0, \mathcal{L}\varphi + (f - \lambda)\varphi \leq 0, \text{ a.e. in } \mathbb{R}^d \right\}. \quad (2.5)$$

Lemma 2.2 *The following hold*

- (i) *For any $r > 0$ the Dirichlet eigensolutions $(\widehat{\Psi}_n, \hat{\lambda}_n)$ in (2.4) have the following stochastic representation*

$$\widehat{\Psi}_n(x) = \mathbb{E}_x \left[e^{\int_0^{\tau_r} [f(X_t) - \hat{\lambda}_n] dt} \widehat{\Psi}_n(X_{\tau_r}) \mathbf{1}_{\{\tau_r < \tau_n\}} \right] \quad \forall x \in B_n \setminus \bar{B}_r, \quad (2.6)$$

for all large enough $n \in \mathbb{N}$.

- (ii) *It holds that $\lambda^*(f) = \hat{\Lambda}(f)$.*
- (iii) *Let Ψ^* be any limit point, as $n \rightarrow \infty$, of the Dirichlet eigensolutions $(\widehat{\Psi}_n, \hat{\lambda}_n)$, and \mathcal{B} be an open ball centered at 0 such that $\lambda^*(f - h) + \sup_{\mathcal{B}^c} |h| < \lambda^*(f) < \infty$ for some bounded function h . Then with τ denoting the first hitting time of \mathcal{B} we have*

$$\Psi^*(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} [f(X_t) - \lambda^*(f)] dt} \Psi^*(X_{\tau}) \mathbf{1}_{\{\tau < \infty\}} \right] \quad \forall x \in \mathcal{B}^c. \quad (2.7)$$

Proof Part (i) follows from [1, Lemma 2.10 (i)].

Turning to part (ii), suppose that $\lambda^*(f)$ is finite. Then it is standard to show that there exists a positive $\Psi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ which satisfies

$$\mathcal{L}\Psi + f\Psi = \lambda^*(f)\Psi \quad \text{a.e. on } \mathbb{R}^d.$$

See [1, 11] for instance. It is then clear that $\lambda^*(f) \geq \hat{\Lambda}(f)$.

To show the converse inequality, suppose that a pair $(\varphi, \lambda) \in \mathcal{W}_{\text{loc}}^{2,d} \times \mathbb{R}$, with $\varphi > 0$, satisfies

$$\mathcal{L}\varphi + (f - \lambda)\varphi \leq 0, \quad \text{and} \quad \lambda \geq \hat{\Lambda}(f). \quad (2.8)$$

We claim that $\lambda^*(f) \leq \lambda$. If not, then we can find a pair $(\widehat{\Psi}_r, \hat{\lambda}_r)$ as in by Lemma 2.1, satisfying (2.4) and $\hat{\lambda}_r > \lambda$. By the Itô–Krylov formula [32, p. 122] we have

$$\varphi(x) \geq \mathbb{E}_x \left[e^{\int_0^{\tau_r} [f(X_t) - \lambda^*(f)] dt} \varphi(X_{\tau_r}) \mathbf{1}_{\{\tau_r < \infty\}} \right]. \quad (2.9)$$

Since φ is positive, (2.8) and (2.9) imply that we can scale it by multiplying with a constant $\kappa > 0$ so that $\kappa\varphi - \widehat{\Psi}_r$ attains its minimum in \bar{B}_r and this minimum value is 0. Combining (2.4) and (2.8) we obtain

$$\mathcal{L}(\kappa\varphi - \widehat{\Psi}_r) - (f - \hat{\lambda}_r)^-(\kappa\varphi - \widehat{\Psi}_r) \leq -(f - \hat{\lambda}_r)^+(\kappa\varphi - \widehat{\Psi}_r) + (-\hat{\lambda}_r + \lambda)\kappa\varphi \leq 0 \quad \text{in } B_r.$$

It then follows by the strong maximum principle [23, Theorem 9.6] that $\kappa\varphi - \widehat{\Psi}_r = 0$ in \bar{B}_r , which is not possible since $\varphi > 0$ on \mathbb{R}^d . This proves the claim. Since λ was arbitrary, this implies that $\hat{\Lambda}(f) \geq \lambda^*(f)$, and thus we have equality.

It remains to prove (2.7). We follow the same argument as in [1, Lemma 2.10]. We fix $\mathcal{B} = B_r$. Letting $n \rightarrow \infty$ in (2.6) and applying Fatou's lemma we obtain

$$\Psi^*(x) \geq \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \lambda^*(f)] dt} \Psi^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right]. \quad (2.10)$$

Thus, with $\tilde{\Psi}^*$ denoting a solution of (2.14) with f replaced by $f - h$, and $\lambda = \lambda^*(f - h)$, we also have

$$\tilde{\Psi}^*(x) \geq \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - h(X_t) - \lambda^*(f - h)] dt} \tilde{\Psi}^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right],$$

which implies that

$$\mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - h(X_t) - \lambda^*(f - h)] dt} \mathbf{1}_{\{\tau < \infty\}} \right] < \infty \quad \forall x \in \mathcal{B}^c, \quad (2.11)$$

since $\tilde{\Psi}^* > 0$ in \mathbb{R}^d . We write (2.6) as

$$\widehat{\Psi}_n(x) \leq \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \hat{\lambda}_n] dt} \Psi^*(X_\tau) \mathbf{1}_{\{\tau < \tau_n\}} \right] + \left(\sup_{\mathcal{B}} |\Psi^* - \widehat{\Psi}_n| \right) \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \hat{\lambda}_n] dt} \mathbf{1}_{\{\tau < \tau_n\}} \right]. \quad (2.12)$$

Note that since $\hat{\lambda}_n \nearrow \lambda^*(f)$, the first term on the right hand side of (2.12) is finite by (2.11) for all large enough n . Let

$$\kappa_n := \left(\inf_{\mathcal{B}} \widehat{\Psi}_n \right)^{-1} \left(\sup_{\mathcal{B}} |\Psi^* - \widehat{\Psi}_n| \right).$$

The second term on the right hand side of (2.12) has the bound

$$\begin{aligned} \left(\sup_{\mathcal{B}} |\Psi^* - \widehat{\Psi}_n| \right) \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \hat{\lambda}_n] dt} \mathbf{1}_{\{\tau < \tau_n\}} \right] &\leq \kappa_n \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \hat{\lambda}_n] dt} \widehat{\Psi}_n(X_\tau) \mathbf{1}_{\{\tau < \tau_n\}} \right] \\ &= \kappa_n \widehat{\Psi}_n(x). \end{aligned}$$

By the convergence of $\widehat{\Psi}_n \rightarrow \Psi^*$ as $n \rightarrow \infty$, uniformly on compact sets, and since $\widehat{\Psi}_n$ is bounded away from 0 in \mathcal{B} , uniformly in $n \in \mathbb{N}$, by Harnack's inequality, we have $\kappa_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, the second term on the right hand side of (2.12) vanishes as $n \rightarrow \infty$. Also, since $\hat{\lambda}_n$ is nondecreasing in n , and $\hat{\lambda}_n \nearrow \lambda^*(f)$, we obtain

$$\mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \hat{\lambda}_n] dt} \Psi^*(X_\tau) \mathbf{1}_{\{\tau < \tau_n\}} \right] \xrightarrow{n \rightarrow \infty} \mathbb{E}_x \left[e^{\int_0^\tau [f(X_t) - \lambda^*(f)] dt} \Psi^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad (2.13)$$

by (2.11) and dominated convergence. Thus taking limits in (2.12) as $n \rightarrow \infty$, and using (2.10) and (2.13) we obtain (2.7). This completes the proof. \square

Combining Lemma 2.2 (ii) and [8, Theorem 1.4] we have the following result.

Corollary 2.1 *There exists a positive $\Psi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p \geq 1$, satisfying*

$$\mathcal{L}\Psi + f\Psi = \lambda\Psi \quad \text{a.e. on } \mathbb{R}^d, \quad (2.14)$$

if and only if $\lambda \geq \lambda^(f)$.*

As also mentioned in the introduction, throughout the rest of the paper, by an eigenpair (Ψ, λ) of \mathcal{L}^f , we mean a positive function $\Psi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ and a scalar $\lambda \in \mathbb{R}$ that satisfy (2.14). In addition, the eigenfunction Ψ is assumed to be normalized as $\Psi(0) = 1$, unless indicated otherwise. When λ is the principal eigenvalue, we refer to (Ψ, λ) as a *principal eigenpair*. Note, that in view of the assumptions on the coefficients, any $\Psi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ which satisfies (2.14) belongs to $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, for all $p \in [1, \infty)$. Therefore, in the interest of notational economy, we refrain from mentioning the function space of solutions Ψ of equations of the form (2.14), and any such solution is meant to be in $\mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$. Moreover, since these are always strong solutions, we often suppress the qualifier ‘a.e.’, and unless a different domain is specified, such equations or inequalities are meant to hold on \mathbb{R}^d .

2.2 The main theorem

A major objective in this paper is to relate the properties of the eigenvalues λ in (2.14) to the recurrence properties of the *twisted process* which is defined as follows. For an eigenfunction Ψ satisfying (2.14) we let $\psi := \log \Psi$. Then we can write (2.14) as

$$\mathcal{L}\psi + \langle \nabla \psi, a \nabla \psi \rangle + f = \lambda. \quad (2.15)$$

The twisted process corresponding to an eigenpair (Ψ, λ) of \mathcal{L}^f is defined by the SDE

$$dY_s = b(Y_s)ds + 2a(Y_s)\nabla \psi(Y_s)ds + \sigma(Y_s)dW_s. \quad (2.16)$$

Since $\psi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p > d$, it follows that $\nabla \psi$ is locally bounded (in fact it is locally Hölder continuous), and therefore (2.16) has a unique strong solution up to its explosion time. We let $\tilde{\mathcal{L}}^\psi$ denote the extended generator of (2.16), and $\tilde{\mathbb{E}}_x^\psi$ the associated expectation operator. The reader might have observed that the twisted process corresponds to Doob’s h -transformation of the operator $\mathcal{L}^{(f-\lambda)}$ with $h = \Psi$.

With Ψ^* denoting a principal eigenfunction, i.e., an eigenfunction associated with $\lambda^*(f)$, we let $\psi^* := \log \Psi^*$, and denote by Y^* the corresponding twisted process. A twisted process corresponding to a principal eigenpair is called a *ground state process*, and the eigenfunction Ψ^* is called a *ground state*.

Recall that $C_0^+(\mathbb{R}^d)$ denotes the collection of all non-trivial, nonnegative, continuous functions which vanish at infinity. We consider the following two properties of $\lambda^*(f)$.

- (P1) Strict monotonicity at f . For some $h \in C_0^+(\mathbb{R}^d)$ we have $\lambda^*(f-h) < \lambda^*(f)$.
- (P2) Strict monotonicity at f on the right. For all $h \in C_0^+(\mathbb{R}^d)$ we have $\lambda^*(f) < \lambda^*(f+h)$.

It follows by the convexity of $f \mapsto \lambda^*(f)$ that (P1) implies (P2).

Later in section 3 we provide sufficient conditions under which (P1) holds. Also, the finiteness of $\lambda^*(f)$ and $\lambda^*(f-h)$ is implicit in (P1). Indeed, since for every positive $\varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$, and $\lambda \in \mathbb{R}$ we have

$$\mathcal{L}\varphi + (f - \lambda - \|h\|_\infty)\varphi \leq \mathcal{L}\varphi + (f - h - \lambda)\varphi \leq \mathcal{L}\varphi + (f - \lambda)\varphi,$$

it follows that $\lambda^*(f-h)$ and $\lambda^*(f)$ are either both finite, or both equal to $\pm\infty$. It is also clear that $\lambda^*(f-h) \leq \lambda^*(f)$ always hold. As shown in Theorem 2.2, (P1) implies that $\lambda^*(f-h) < \lambda^*(f)$ for all $h \in C_0^+(\mathbb{R}^d)$.

We introduce the following definition of exponential ergodicity which we often use.

Definition 2.2 (exponential ergodicity) The process X governed by (1.2) is said to be exponentially ergodic if for some compact set \mathcal{B} and $\delta > 0$ we have

$$\mathbb{E}_x[e^{\delta \tau(\mathcal{B}^c)}] < \infty \quad \text{for all } x \in \mathcal{B}^c.$$

The bulk of the main results of this section can be summarized as follows.

Theorem 2.1 *Under (H1), the following hold:*

(a) *A ground state process is recurrent if and only if $\lambda^*(f)$ is strictly monotone at f on the right, in which case the principal eigenvalue $\lambda^*(f)$ is also simple, and the ground state Ψ^* satisfies*

$$\Psi^*(x) = \mathbb{E}_x \left[e^{\int_0^{\tau_r} [f(X_s) - \lambda^*(f)] ds} \Psi^*(X_{\tau_r}) \mathbf{1}_{\{\tau_r < \infty\}} \right] \quad \forall x \in \bar{B}_r^c, \quad \forall r > 0. \quad (2.17)$$

(b) *The ground state process is exponentially ergodic if and only if $\lambda^*(f)$ is strictly monotone at f .*

(c) *If $\lambda > \lambda^*(f)$, the twisted process (2.16) corresponding to any solution ψ of (2.15) is transient.*

Proof Part (a) follows by Lemma 2.7, Theorem 2.3, and Corollary 2.3. Part (b) is the statement of Theorem 2.2, while part (c) is shown in Lemma 2.6. \square

Theorem 2.1 should be compared with the results in [26, Theorem 2.2] and [30, Theorem 3.2 and 3.7]. The results in [26, 30] are obtained under a stronger hypothesis (same as (3.12) below) and for sufficiently regular coefficients. For a similar result in a bounded domain we refer the reader to [38, Theorem 4.2.4], where results are obtained for a certain class of operators with regular coefficients.

We remark that (P1) does not imply that the underlying process in (2.1) is recurrent. Indeed consider a one-dimensional diffusion with $b(x) = \frac{3}{2}x$, and $\sigma = 1$, and let $f(x) = x^2$. Then (2.14) holds with $\Psi(x) = e^{-x^2}$ and $\lambda = -1$. But $b(x) + 2a\nabla\psi = -\frac{1}{2}x$, so the twisted process is exponentially ergodic, while the original diffusion is transient.

The proof of Theorem 2.1 is divided in several lemmas which also contain results of independent interests. These occupy the next section.

2.3 Proof of Theorem 2.1 and other results

In the sequel, we often use the following finite time representation. This also appears in [1, Lemma 2.4] but in a slightly different form. Let $\tau_\infty := \lim_{n \rightarrow \infty} \tau_n$ where τ_n denotes the exit time from the ball B_n . Recall that if (Ψ, λ) is an eigenpair of \mathcal{L}^f , and $\psi = \log \Psi$, then $\widetilde{\mathbb{E}}_x^\psi$ denotes the expectation operator associated with the twisted process Y in (2.16).

Lemma 2.3 *If (Ψ, λ) is an eigenpair of \mathcal{L}^f , then*

$$\Psi(x) \widetilde{\mathbb{E}}_x^\psi [g(Y_T) \Psi^{-1}(Y_T) \mathbf{1}_{\{T < \tau_\infty\}}] = \mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda] dt} g(X_T) \right] \quad \forall T > 0, \quad \forall x \in \mathbb{R}^d, \quad (2.18)$$

and for any function $g \in C_c(\mathbb{R}^d)$, where Y is the corresponding twisted process defined by (2.16).

Proof The equation in (2.18) can be obtained by applying the Cameron–Martin–Girsanov theorem [34, p. 225]. Since ψ and f are not bounded, we need to localize the martingale. We use the first exit times from B_n , i.e., τ_n as localization times. Due to (A2) it is well-known that $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ \mathbb{P}_x -a.s. Applying the Itô–Krylov formula to (2.16) we obtain

$$\begin{aligned} \psi(X_{T \wedge \tau_n}) - \psi(x) &= \int_0^{T \wedge \tau_n} \mathcal{L} \psi(X_s) ds + \int_0^{T \wedge \tau_n} \langle \nabla \psi(X_s), \sigma(X_s) dW_s \rangle \\ &= \int_0^{T \wedge \tau_n} \left(\lambda - f(X_s) - \langle \nabla \psi, a \nabla \psi \rangle(X_s) \right) ds + \int_0^{T \wedge \tau_n} \langle \nabla \psi(X_s), \sigma(X_s) dW_s \rangle. \end{aligned} \quad (2.19)$$

Let g be any nonnegative, continuous function with compact support. Then from (2.19) we obtain

$$\begin{aligned} \mathbb{E}_x \left[e^{\int_0^{T \wedge \tau_n} (f(X_s) - \lambda) ds} g(X_{T \wedge \tau_n}) \right] &= \mathbb{E}_x \left[g(X_{T \wedge \tau_n}) \exp \left(-\psi(X_{T \wedge \tau_n}) + \psi(x) \right) \right. \\ &\quad \left. + \int_0^{T \wedge \tau_n} \langle \nabla \psi(X_s), \sigma(X_s) dW_s \rangle - \int_0^{T \wedge \tau_n} \langle \nabla \psi, a \nabla \psi \rangle(X_s) ds \right] \end{aligned}$$

$$= \Psi(x) \widetilde{\mathbb{E}}_x^\Psi \left[g(Y_{T \wedge \tau_n}) \Psi^{-1}(Y_{T \wedge \tau_n}) \right], \quad (2.20)$$

where in the last line we use Girsanov's theorem. Given any bounded ball \mathcal{B} , by Itô's formula and Fatou's lemma, we obtain from (2.14) that

$$\mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda] dt} \mathbf{1}_{\mathcal{B}}(X_T) \right] \leq \left(\inf_{\mathcal{B}} \Psi \right)^{-1} \Psi(x) \quad \forall T > 0, \quad \forall x \in \mathbb{R}^d. \quad (2.21)$$

Therefore writing,

$$\mathbb{E}_x \left[e^{\int_0^{T \wedge \tau_n} (f(X_s) - \lambda) ds} g(X_{T \wedge \tau_n}) \right] = \mathbb{E}_x \left[e^{\int_0^{\tau_n} (f(X_s) - \lambda) ds} g(X_{\tau_n}) \mathbf{1}_{\{T \geq \tau_n\}} \right] + \mathbb{E}_x \left[e^{\int_0^T (f(X_s) - \lambda) ds} g(X_T) \mathbf{1}_{\{T < \tau_n\}} \right],$$

we deduce that the first term on the right hand side equal to 0 for all n large since g is compactly supported and, while the second term converges as $n \rightarrow \infty$ to the right hand side of (2.18) by (2.21) and dominated convergence. In addition, since g has compact support, the term inside the expectation in the right hand side of (2.20) is bounded uniformly in n . Since also $\widetilde{\mathbb{E}}_x^\Psi [g(Y_{\tau_n}) \Psi^{-1}(Y_{\tau_n})] = 0$ for all sufficiently large n , letting $n \rightarrow \infty$ in (2.20), we obtain

$$\begin{aligned} \mathbb{E}_x \left[e^{\int_0^T (f(X_s) - \lambda) ds} g(X_T) \right] &= \lim_{n \rightarrow \infty} \Psi(x) \widetilde{\mathbb{E}}_x^\Psi \left[g(Y_T) \Psi^{-1}(Y_T) \mathbf{1}_{\{T < \tau_n\}} \right] \\ &= \Psi(x) \widetilde{\mathbb{E}}_x^\Psi \left[g(Y_T) \Psi^{-1}(Y_T) \mathbf{1}_{\{T < \tau_\infty\}} \right] \quad \forall T > 0. \end{aligned}$$

This proves (2.18). \square

Recall that $\tau_\infty := \lim_{n \rightarrow \infty} \tau_n$. An immediate corollary to Lemma 2.3 is the following.

Corollary 2.2 *With (Ψ, λ) as in Lemma 2.3, we have*

$$\Psi(x) \widetilde{\mathbb{P}}_x^\Psi (T < \tau_\infty) = \mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda] dt} \Psi(X_T) \right] \quad \forall T > 0, \quad \forall x \in \mathbb{R}^d.$$

Proof Choose a sequence of cut-off functions g_n that approximates unity from below. Then (2.18) holds for g replaced by $g_n \Psi$. Therefore the result follows by letting $n \rightarrow \infty$ and applying the monotone convergence theorem. \square

We are now ready to prove uniqueness of the principal eigenfunction.

Lemma 2.4 *Under (P1) there exists a unique ground state Ψ^* of \mathcal{L}^f , i.e., a positive $\Psi^* \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$, $\Psi^*(0) = 1$, which solves*

$$\mathcal{L}\Psi^* + f\Psi^* = \lambda^*(f)\Psi^*. \quad (2.22)$$

Proof Let Ψ^* be a solution of (2.22) obtained as a limit of $\widehat{\Psi}_r$ (see Lemma 2.2). Thus by Lemma 2.2 (iii) we can find a ball \mathcal{B} such that

$$\Psi^*(x) = \mathbb{E}_x \left[e^{\int_0^\tau (f(X_s) - \lambda^*(f)) ds} \Psi^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad x \in \mathcal{B}^c. \quad (2.23)$$

Suppose that $\tilde{\Psi}$ is another principal eigenfunction of \mathcal{L}^f . By Itô–Krylov's formula and Fatou's lemma we obtain

$$\tilde{\Psi}(x) \geq \mathbb{E}_x \left[e^{\int_0^\tau (f(X_s) - \lambda^*(f)) ds} \tilde{\Psi}(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad x \in \mathcal{B}^c. \quad (2.24)$$

Now we multiply Ψ^* with a suitable positive constant so that $\tilde{\Psi}$ touches Ψ^* from above in \mathcal{B} . Thus from (2.23) and (2.24) we deduce that $\tilde{\Psi} - \Psi^*$ is nonnegative in \mathbb{R}^d , and its minimum is 0 and attained in \mathcal{B} . On the other hand, we have

$$\mathcal{L}(\tilde{\Psi} - \Psi^*) - (f - \lambda^*(f))(\tilde{\Psi} - \Psi^*) = -(f - \lambda^*(f))^+(\tilde{\Psi} - \Psi^*) \leq 0.$$

Thus $\tilde{\Psi} - \Psi^* = 0$ by the strong maximum principle [23, Theorem 9.6], and this proves the result. \square

We next show that (P1) implies exponential ergodicity for Y^* .

Lemma 2.5 Assume (P1). Let Ψ^* be the principal eigenfunction of \mathcal{L}^f , and $\psi^* = \log \Psi^*$. Then the ground state process Y^* governed by

$$dY_s^* = b(Y_s^*) ds + 2a(Y_s^*) \nabla \psi^*(Y_s^*) ds + \sigma(Y_s^*) dW_s, \quad (2.25)$$

is exponentially ergodic. In particular, Y^* is positive recurrent.

Proof We first show that the finite time representation of Ψ^* holds. Let $\tilde{\lambda}^* := \lambda^*(f - h)$, and \mathcal{B} be a ball as in Lemma 2.2 (iii). Recall that $\hat{\lambda}_n \rightarrow \lambda^*(f)$ as $n \rightarrow \infty$, and therefore, we have $\hat{\lambda}_n > \tilde{\lambda}^* + \sup_{\mathcal{B}^c} |h|$ for all sufficiently large n . Consider the following equations

$$\begin{aligned} \mathcal{L}\hat{\Psi}_n + f\hat{\Psi}_n &= \hat{\lambda}_n \hat{\Psi}_n, \quad \hat{\Psi}_n > 0, \quad \hat{\Psi}_n(0) = 1, \quad \hat{\Psi}_n = 0 \text{ on } \partial B_n, \\ \mathcal{L}\tilde{\Psi}^* + (f - h)\tilde{\Psi}^* &= \tilde{\lambda}^* \tilde{\Psi}^*, \quad \tilde{\Psi}^*(0) = 1. \end{aligned}$$

Choose n large enough so that $\mathcal{B} \subset B_n$. We can scale $\tilde{\Psi}^*$, by multiplying it with a positive constant, so that $\tilde{\Psi}^*$ touches $\hat{\Psi}_n$ from above. Next we show that it can only touch $\hat{\Psi}_n$ in \mathcal{B} . Note that in $B_n \setminus \mathcal{B}$ we have

$$\mathcal{L}(\tilde{\Psi}^* - \hat{\Psi}_n) - (f - h - \tilde{\lambda}^*)^-(\tilde{\Psi}^* - \hat{\Psi}_n) = -(f - h - \tilde{\lambda}^*)^+(\tilde{\Psi}^* - \hat{\Psi}_n) - (\hat{\lambda}_n - \tilde{\lambda}^* - h)\hat{\Psi}_n \leq 0.$$

Therefore, by the strong maximum principle, if $(\tilde{\Psi}^* - \hat{\Psi}_n)$ attains its minimum in $B_n \setminus \mathcal{B}$, then $(\tilde{\Psi}^* - \hat{\Psi}_n) = 0$ in B_n , which is not possible. Thus $\tilde{\Psi}^*$ touches $\hat{\Psi}_n$ in \mathcal{B} . Thus, applying Harnack's inequality we can find a constant κ_1 such that $\kappa_1 \tilde{\Psi}^* \geq \hat{\Psi}_n$ for all n large. On the other hand, by Itô–Krylov's formula and Fatou's lemma we know that

$$\mathbb{E}_x \left[e^{\int_0^T [f(X_s) - h(X_s) - \tilde{\lambda}^*] ds} \tilde{\Psi}^*(X_T) \right] \leq \tilde{\Psi}^*(x) \quad \forall T > 0. \quad (2.26)$$

Applying Itô–Krylov's formula to (2.4) we have

$$\hat{\Psi}_n(x) = \mathbb{E}_x \left[e^{\int_0^T [f(X_s) - \hat{\lambda}_n] ds} \hat{\Psi}_n(X_T) \mathbf{1}_{\{T < \tau_n\}} \right],$$

and letting $n \rightarrow \infty$, using (2.26) and the dominated convergence theorem, we obtain

$$\Psi^*(x) = \mathbb{E}_x \left[e^{\int_0^T [f(X_s) - \lambda^*] ds} \Psi^*(X_T) \right],$$

where Ψ^* is the unique solution of (2.22). This proves the finite time representation. Thus it follows from Corollary 2.2 that (2.25) is regular, i.e., $\tilde{\mathbb{P}}_x^{\Psi^*}(\tau_\infty < \infty) = 0$.

Next we define

$$\Phi = \frac{\tilde{\Psi}^*}{\Psi^*}.$$

Then a straightforward calculation shows that

$$\tilde{\mathcal{L}}^{\Psi^*} \Phi = \mathcal{L}\Phi + 2\nabla \Phi \cdot a \nabla \psi^* = (\lambda^*(f - h) - \lambda^*(f) + h)\Phi \leq C \mathbf{1}_{\mathcal{B}} - \varepsilon \Phi \quad (2.27)$$

for some positive constants C and ε . Recall \mathcal{B} from Lemma 2.2 (iii). It is easy to see from (2.7) that

$$\Phi(x) \geq \frac{\min_{\mathcal{B}} \tilde{\Psi}^*}{\max_{\mathcal{B}} \Psi^*} \frac{\mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f(X_t) - h(X_t) - \lambda^*(f - h)] dt} \mathbf{1}_{\{\tilde{\tau} < \infty\}} \right]}{\mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f(X_t) - \lambda^*(f)] dt} \mathbf{1}_{\{\tilde{\tau} < \infty\}} \right]} \geq \frac{\min_{\mathcal{B}} \tilde{\Psi}^*}{\max_{\mathcal{B}} \Psi^*} \quad \forall x \in \mathcal{B}^c.$$

Thus Φ is uniformly bounded from below by a positive constant. Since Y^* in (2.25) is regular, the Foster–Lyapunov inequality in (2.27) implies that Y^* is exponentially ergodic. \square

We denote the invariant measure of (2.25) by μ^* . The following lemma shows that the twisted process is transient for any $\lambda > \lambda^*(f)$.

Lemma 2.6 *Let Ψ be an eigenfunction of \mathcal{L}^f for an eigenvalue $\lambda > \lambda^*(f)$. Then the corresponding twisted process Y is transient.*

Proof Let $\psi = \log \Psi$. If $\tilde{\mathbb{P}}_x^\psi(\tau_\infty < \infty) > 0$, then there is nothing to prove. So we assume the contrary. Hence from Lemma 2.3 we have

$$\Psi(x) \tilde{\mathbb{E}}_x^\Psi[g(Y_T) \Psi^{-1}(Y_T)] = \mathbb{E}_x\left[e^{\int_0^T [f(X_t) - \lambda] dt} g(X_T)\right] \quad \forall T > 0, \quad (2.28)$$

for any continuous g with compact support. Let g be such a function.

By the Itô–Krylov formula and Fatou’s lemma, we have

$$\begin{aligned} \mathbb{E}_x\left[e^{\int_0^T [f(X_t) - \lambda^*(f)] dt} g(X_T)\right] &\leq \left(\sup_{\mathbb{R}^d} \frac{g}{\Psi^*}\right) \mathbb{E}_x\left[e^{\int_0^T [f(X_t) - \lambda^*(f)] dt} \Psi^*(X_T)\right] \\ &\leq \left(\sup_{\mathbb{R}^d} \frac{g}{\Psi^*}\right) \Psi^*(x) \end{aligned}$$

Thus, for $\delta = \lambda - \lambda^*(f) > 0$, we obtain

$$\mathbb{E}_x\left[e^{\int_0^T [f(X_t) - \lambda] dt} g(X_T)\right] \leq \left(\sup_{\mathbb{R}^d} \frac{g}{\Psi^*}\right) e^{-\delta T} \Psi^*(x), \quad T > 0. \quad (2.29)$$

Combining (2.28) and (2.29) we obtain

$$\begin{aligned} \frac{1}{\sup_{\text{support}(g)} \Psi} \int_0^\infty \tilde{\mathbb{E}}_x^\Psi[g(Y_t)] dt &\leq \int_0^\infty \tilde{\mathbb{E}}_x^\Psi\left[\frac{g(Y_t)}{\Psi(Y_t)}\right] dt \\ &\leq \frac{1}{\delta} \left(\sup_{\mathbb{R}^d} \frac{g}{\Psi^*}\right) \Psi^*(x) \Psi^{-1}(x). \end{aligned}$$

Therefore Y is transient. □

Theorem 2.2 *The following are equivalent.*

- (i) *The twisted process Y^* , defined by (2.25), corresponding to some principal eigenpair $(\Psi^*, \lambda^*(f))$ is exponentially ergodic.*
- (ii) *It holds that $\lambda^*(f - h) < \lambda^*(f)$ for all $h \in C_0^+(\mathbb{R}^d)$.*
- (iii) *It holds that $\lambda^*(f - h) < \lambda^*(f)$ for some $h \in C_0^+(\mathbb{R}^d)$.*

Proof (iii) \Rightarrow (i) follows from Lemma 2.5.

We show that (i) \Rightarrow (ii). If Y^* is exponentially ergodic, then there exists a ball \mathcal{B} and $\delta > 0$ such that

$$\tilde{\mathbb{E}}_x^{\Psi^*}[e^{\delta \check{\tau}}] < \infty, \quad \check{\tau} = \check{\tau}(\mathcal{B}^c).$$

From the calculation of Lemma 2.3 we obtain that for $g \in C_c(\mathbb{R}^d)$

$$\mathbb{E}_x\left[e^{\int_0^{T \wedge \check{\tau}} (f(X_s) - \lambda^*(f) + \delta) ds} g(X_{T \wedge \check{\tau}}) \Psi^*(X_{T \wedge \check{\tau}})\right] = \Psi^*(x) \tilde{\mathbb{E}}_x^{\Psi^*}\left[e^{\delta(T \wedge \check{\tau})} g(Y_{T \wedge \check{\tau}}^*)\right] \quad \forall T > 0.$$

Choose a sequence of g_m increasing to 1, and let $m \rightarrow \infty$ first, and then $T \rightarrow \infty$, using Fatou’s lemma to obtain

$$\mathbb{E}_x\left[e^{\int_0^{\check{\tau}} (f(X_s) - \lambda^*(f) + \delta) ds} \mathbf{1}_{\{\check{\tau} < \infty\}}\right] < \infty, \quad x \in \mathcal{B}^c. \quad (2.30)$$

Let $h \in C_0^+(\mathbb{R}^d)$. Since h is bounded it is easy to see that $\lambda^*(f - h)$ is finite. Let $\tilde{f} := f - h$, and $(\tilde{\Psi}^*, \lambda^*(\tilde{f}))$ be a solution of

$$\mathcal{L}\tilde{\Psi}^* + (f - h)\tilde{\Psi}^* = \lambda^*(\tilde{f})\tilde{\Psi}^*, \quad \tilde{\Psi}^* > 0, \quad (2.31)$$

that is obtained as a limit of Dirichlet eigensolutions as in [Lemma 2.2](#). Now if $\lambda^*(\tilde{f}) = \lambda^*(f)$, then in view of [\(2.30\)](#) and the calculations in the proof of [Lemma 2.2](#) (iii) we have

$$\tilde{\Psi}^*(x) = \mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f(X_t) - h(X_t) - \lambda^*(\tilde{f})] dt} \tilde{\Psi}^*(X_{\tilde{\tau}}) \mathbf{1}_{\{\tilde{\tau} < \infty\}} \right] \quad \forall x \in \mathcal{B}^c. \quad (2.32)$$

Applying the Itô–Krylov formula and Fatou’s lemma to [\(2.22\)](#) we have

$$\Psi^*(x) \geq \mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} (f(X_s) - \lambda^*(f)) ds} \Psi^*(X_{\tilde{\tau}}) \mathbf{1}_{\{\tilde{\tau} < \infty\}} \right], \quad x \in \mathcal{B}^c. \quad (2.33)$$

It follows by [\(2.32\)](#) and [\(2.33\)](#) that we can multiply Ψ^* with suitable positive constant so that $\Psi^* - \tilde{\Psi}^*$ attains a minimum of 0 in \mathcal{B} . On the other hand, from [\(2.22\)](#) and [\(2.31\)](#) we have

$$\mathcal{L}(\Psi^* - \tilde{\Psi}^*) - (f - \lambda^*(f))^- (\Psi^* - \tilde{\Psi}^*) = -(f - \lambda^*(f))^+ (\Psi^* - \tilde{\Psi}^*) - h \tilde{\Psi}^* \leq 0. \quad (2.34)$$

Thus by strong maximum principle we have $\Psi^* = \tilde{\Psi}^*$. This, in turn, implies that $h \tilde{\Psi}^* = 0$ by [\(2.34\)](#). But this is not possible. Hence we have $\lambda^*(\tilde{f}) < \lambda^*(f)$, and the proof is complete. \square

Recall that $C_c^+(\mathbb{R}^d)$ is the subset of $C_0^+(\mathbb{R}^d)$ containing functions with compact support. We define the Green’s measure G_λ , $\lambda \in \mathbb{R}$, by

$$G_\lambda(g) = \mathbb{E}_0 \left[\int_0^\infty e^{\int_0^t [f(X_s) - \lambda] ds} g(X_t) dt \right] \quad \text{for all } g \in C_c^+(\mathbb{R}^d).$$

The density of the Green’s measure with respect to the Lebesgue measure is called the Green’s function. Existence of a Green’s function (and Green’s measure) is used by Pinsky [\[38, Chapter 4.3\]](#) in his definition of the generalized principal eigenvalue of \mathcal{L}^f . A number $\lambda \in \mathbb{R}$ is said to be *subcritical* if G_λ possesses a density, it is said to be *critical* if it is not subcritical but $\mathcal{L}^{f-\lambda} V = 0$ has a positive solution V , and λ is said to be *supercritical* if it is neither subcritical nor critical.

The lemma which follows is an extension of [\[38, Theorem 4.3.4\]](#) where, under a regularity assumption on the coefficients, it is shown that a critical eigenvalue λ is always simple. It should be noted that the result below establishes several equivalent relation with criticality of λ .

Lemma 2.7 *The following are equivalent.*

- (i) *The twisted process Y corresponding to the eigenpair (Ψ, λ) is recurrent.*
- (ii) *$G_\lambda(g)$ is infinite for some $g \in C_c^+(\mathbb{R}^d)$.*
- (iii) *For some open ball \mathcal{B} , and with $\tilde{\tau} = \tilde{\tau}(\mathcal{B})$, we have*

$$\Psi(x) = \mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f(X_s) - \lambda] ds} \Psi(X_{\tilde{\tau}}) \mathbf{1}_{\{\tilde{\tau} < \infty\}} \right], \quad x \in \bar{\mathcal{B}}^c,$$

where Ψ is an eigenfunction corresponding to the eigenvalue λ .

In addition, in (ii)–(iii) “some” may be replaced by “all”, and if any one of (i)–(iii) holds, then λ is a simple eigenvalue.

Proof The argument of this proof is inspired from [\[1, Theorem 2.8\]](#). By [Corollary 2.1](#) we have $\lambda \geq \lambda^*(f)$. Assume that (i) holds for some $\lambda \geq \lambda^*(f)$. Let (Ψ, λ) be an eigenpair of \mathcal{L}^f . Then for any $g \in C_c^+(\mathbb{R}^d)$ we have from [Lemma 2.3](#) that

$$\Psi(x) \tilde{\mathbb{E}}_x^\Psi [g(Y_T) \Psi^{-1}(Y_T)] = \mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda] dt} g(X_T) \right] \quad \forall T > 0, \quad (2.35)$$

Now if Y is recurrent then

$$\int_0^\infty \tilde{\mathbb{E}}_x^\Psi [g(Y_t) \Psi^{-1}(Y_t)] dt = \infty.$$

Combining this with [\(2.35\)](#) we have $G_\lambda(g) = \infty$. Hence (ii) follows.

Next suppose that (ii) holds, i.e., $G_\lambda(g) = \infty$ for some $g \in C_c^+(\mathbb{R}^d)$ and $\lambda \geq \lambda^*$. Applying the Itô–Krylov formula to $\mathcal{L}\Psi + (f - \lambda)\Psi = 0$, we have

$$\mathbb{E}_x \left[e^{\int_0^t [f(X_s) - \lambda] ds} \mathbf{1}_{\mathcal{B}}(X_t) \right] \leq \frac{1}{\min_{\mathcal{B}} \Psi} \mathbb{E}_x \left[e^{\int_0^t [f(X_s) - \lambda] ds} \Psi(X_t) \right] \leq \frac{1}{\min_{\mathcal{B}} \Psi} \Psi(x), \quad \forall t \geq 0, \quad (2.36)$$

for any bounded ball \mathcal{B} . Fix $g \in C_c^+(\mathbb{R}^d)$, and $\alpha > 0$. Define $F_\alpha(x) = f(x) - \lambda - \alpha$ and

$$\Gamma_\alpha = \mathbb{E}_0 \left[\int_0^\infty e^{\int_0^t F_\alpha(X_s) ds} g(X_t) dt \right].$$

From (2.36) we have $\Gamma_\alpha < \infty$ for all $\alpha > 0$. Moreover, $\Gamma_\alpha \rightarrow \infty$ as $\alpha \rightarrow 0$ by hypothesis. Choose n_0 large enough so that $\text{support}(g) \subset B_{n_0}$. Following [1, Theorem 2.8] we consider the positive solution $\varphi_{\alpha,n} \in \mathcal{W}_{\text{loc}}^{2,p}(B_n) \cap C(\bar{B}_n)$ of

$$\mathcal{L}\varphi_{\alpha,n} + F_\alpha \varphi_{\alpha,n} = -\Gamma_\alpha^{-1}g \quad \text{in } B_n, \quad \varphi_{\alpha,n} = 0 \quad \text{on } \partial B_n, \quad (2.37)$$

for $n \geq n_0$. Since for every fixed n we have

$$\mathbb{E}_x \left[e^{\int_0^T F_\alpha(X_s) ds} \varphi_{\alpha,n}(X_T) \mathbf{1}_{\{T \leq \tau_n\}} \right] \leq \left(\max_{B_n} \varphi_{\alpha,n} \right) e^{-\alpha T} \mathbb{E}_x \left[e^{\int_0^T F_0(X_s) ds} \mathbf{1}_{B_n}(X_T) \mathbf{1}_{\{T \leq \tau_n\}} \right] \xrightarrow{T \rightarrow \infty} 0$$

by (2.36), applying Itô–Krylov’s formula to (2.37), we obtain by [1, Theorem 2.8] that

$$\varphi_{\alpha,n}(0) = \Gamma_\alpha^{-1} \mathbb{E}_0 \left[\int_0^{\tau_n} e^{\int_0^t F_\alpha(X_s) ds} g(X_t) dt \right] \leq \Gamma_\alpha^{-1} \Gamma_\alpha = 1. \quad (2.38)$$

Since Γ_α^{-1} is bounded uniformly on $\alpha \in (0, 1)$ by hypothesis, we can apply Harnack’s inequality for a class of superharmonic functions in [3] to conclude that $\{\varphi_{\alpha,n}, n \in \mathbb{N}\}$ is locally bounded, and therefore also uniformly bounded in $\mathcal{W}_{\text{loc}}^{2,p}(B_R)$, $p > d$, for any $R > 0$. Thus, we have that $\varphi_{\alpha,n} \rightarrow \varphi_\alpha$ weakly in $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$ along some subsequence, and that φ_α satisfies

$$\mathcal{L}\varphi_\alpha + F_\alpha \varphi_\alpha = -\Gamma_\alpha^{-1}g \quad \text{in } \mathbb{R}^d \quad (2.39)$$

by (2.37). Let \mathcal{B} be a closed ball around 0 such that $\text{support}(g) \subset \mathcal{B}$. Applying Itô–Krylov’s formula to (2.37) we obtain

$$\varphi_{\alpha,n}(x) = \mathbb{E}_x \left[e^{\int_0^{\tau \wedge T} F_\alpha(X_s) ds} \varphi_{\alpha,n}(X_{\tau \wedge T}) \mathbf{1}_{\{\tau \wedge T < \tau_n\}} \right], \quad x \in B_n \setminus \mathcal{B}, \quad \forall T > 0,$$

with $\tau = \tau(\mathcal{B}^c)$. As in the derivation of (2.38), using (2.36) and a similar argument we obtain

$$\varphi_{\alpha,n}(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} F_\alpha(X_s) ds} \varphi_{\alpha,n}(X_\tau) \mathbf{1}_{\{\tau < \tau_n\}} \right], \quad x \in B_n \setminus \mathcal{B}. \quad (2.40)$$

Letting $n \rightarrow \infty$ along some subsequence, and arguing as above, we obtain a function φ_α which satisfies (2.39) and

$$\varphi_\alpha(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} F_\alpha(X_s) ds} \varphi_\alpha(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad x \in \mathcal{B}^c, \quad (2.41)$$

where (2.41) follows from (2.40). From (2.38) we have $\varphi_\alpha(0) = 1$ for all $\alpha \in (0, 1)$. Now applying Harnack’s inequality once again and letting $\alpha \searrow 0$, we deduce that φ_α converges weakly in $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p > d$, to some positive function Ψ which satisfies

$$\mathcal{L}\Psi + F_0\Psi = 0 \quad \text{in } \mathbb{R}^d,$$

and

$$\Psi(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} F_0(X_s) ds} \Psi(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad x \in \mathcal{B}^c.$$

This implies (iii).

Lastly, suppose that (iii) holds. In other words, there exists an eigenpair (Ψ, λ) and an open ball \mathcal{B} such that

$$\Psi(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} F_0(X_s) ds} \Psi(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad x \in \mathcal{B}^c. \quad (2.42)$$

We first show that λ is a simple eigenvalue, which implies that there is a unique twisted process Y corresponding to λ . To establish the simplicity of λ consider another eigenpair $(\tilde{\Psi}, \lambda)$ of \mathcal{L}^f . By the Itô–Krylov formula we obtain

$$\tilde{\Psi}(x) \geq \mathbb{E}_x \left[e^{\int_0^\tau F_0(X_s) ds} \tilde{\Psi}(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \quad x \in \mathcal{B}^c.$$

Thus using (2.42) and an argument similar to Lemma 2.4 we can show that $\Psi = \tilde{\Psi}$. Then (iii) \Rightarrow (i) follows from [1, Lemma 2.6].

Uniqueness of the eigenfunction Ψ follows from the stochastic representation in (2.42) and the proof of (iii) \Rightarrow (i). \square

As an immediate corollary to Lemmas 2.6 and 2.7 we have the following.

Corollary 2.3 *Let (Ψ, λ) be an eigenpair of \mathcal{L}^f which satisfies*

$$\Psi(x) = \mathbb{E}_x \left[e^{\int_0^\tau [f(X_s) - \lambda] ds} \Psi(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] \quad \forall x \in \mathcal{B}^c.$$

for some bounded open ball \mathcal{B} in \mathbb{R}^d . Then $\lambda = \lambda^(f)$, and it is a simple eigenvalue.*

We next state the following result which is a generalization of [38, Theorem 4.7.1] in \mathbb{R}^d . The result in [38, Theorem 4.7.1] has been obtained for bounded and regular coefficients in bounded domains. It is shown in [38] that for smooth bounded domains, the Green's measure is not defined at the critical value λ^* [38, Theorem 3.2]. But by Theorem 2.3 below we see that this is not the case in unbounded domain. In fact, by [38, Theorem 4.3.2] λ^* could be either subcritical or critical in the sense of Pinsky. We show that the criticality of λ^* is equivalent to the strict monotonicity of $\lambda^*(f)$ on the right, i.e., $\lambda^*(f) < \lambda^*(f+h)$ for all $h \in C_0^+(\mathbb{R}^d)$.

Theorem 2.3 *A ground state process is recurrent if and only if $\lambda^*(f) < \lambda^*(f+h)$ for all $h \in C_0^+(\mathbb{R}^d)$.*

Proof Suppose first that a ground state process corresponding to $\lambda^*(f)$ is recurrent. Then $G_{\lambda^*}(g) = \infty$ for all $g \in C_c^+(\mathbb{R}^d)$ by Lemma 2.7. Let $\tilde{f} = f+h$ and $\tilde{\lambda}^* := \lambda^*(f+h)$. Suppose that $\lambda^* = \tilde{\lambda}^*$. Let $\tilde{\Psi}$ be a principal eigenfunction of $\mathcal{L}^{\tilde{f}}$, i.e.,

$$\mathcal{L}\tilde{\Psi} + \tilde{f}\tilde{\Psi} = \tilde{\lambda}^*\tilde{\Psi}. \quad (2.43)$$

Writing (2.43) as

$$\mathcal{L}\tilde{\Psi} + (f - \lambda^*)\tilde{\Psi} = -h\tilde{\Psi},$$

and applying the Itô–Krylov's formula, followed by Fatou's lemma, we obtain

$$\mathbb{E}_x \left[e^{\int_0^T [f(X_s) - \lambda^*] ds} \tilde{\Psi}(X_T) \right] + \int_0^T \mathbb{E}_x \left[e^{\int_0^t [f(X_s) - \lambda^*] ds} h(X_t) \tilde{\Psi}(X_t) \right] dt \leq \tilde{\Psi}(x),$$

which contradicts the property that $G_{\lambda^*}(g) = \infty$ for all $g \in C_c^+(\mathbb{R}^d)$. Therefore, $\lambda^*(f) < \lambda^*(f+h)$ for all $h \in C_0$.

To prove the converse we assume that Y^* is transient. Then for $g \in C_c^+(\mathbb{R}^d)$ with $B_1 \subset \text{support}(g)$ we have $G_{\lambda^*}(g) < \infty$. Following the arguments in the proof of (ii) \Rightarrow (iii) in Lemma 2.7, we obtain a positive Φ satisfying

$$\mathcal{L}\Phi + (f - \lambda^*)\Phi = -\Gamma_0^{-1}g. \quad (2.44)$$

Let $\varepsilon = \Gamma_0^{-1} \min_{B_1} \frac{g}{\Phi}$. Then from (2.44) we have

$$\mathcal{L}\Phi + (f + \varepsilon \mathbf{1}_{B_1} - \lambda^*)\Phi \leq 0.$$

This implies that $\lambda^*(f + \varepsilon \mathbf{1}_{B_1}) \leq \lambda^*(f)$ by Lemma 2.2 (ii), which in turn implies that $\lambda^*(f + \varepsilon \mathbf{1}_{B_1}) = \lambda^*(f)$. Therefore, if $\lambda^*(f) < \lambda^*(f+h)$ for all $h \in C_0$, then Y^* has to be recurrent. This completes the proof. \square

It is well known that a (null) recurrent diffusion $\{X_t\}$ with locally uniformly elliptic and Lipschitz continuous a , and locally bounded measurable drift, admits a σ -finite invariant probability measure ν which is a Radon measure on the Borel σ -field of \mathbb{R}^d [25]. This measure is equivalent to the Lebesgue measure and is unique up to a multiplicative constant. Theorem 8.1 in [25] is that if g and h are real-valued functions which are integrable with respect to the measure ν then

$$\mathbb{P}_x \left(\lim_{T \rightarrow \infty} \frac{\int_0^T g(X_t) dt}{\int_0^T h(X_t) dt} = \frac{\int_{\mathbb{R}^d} g(x) \nu(dx)}{\int_{\mathbb{R}^d} h(x) \nu(dx)} \right) = 1. \quad (2.45)$$

Suppose $g: \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a non-trivial function. Select h as the indicator function of some open ball. Then it is well known that the expectation of $Y_t^h := \int_0^t h(X_s) ds$ tends to ∞ as $t \rightarrow \infty$. Adopt the analogous notation Y_t^g , and let $\alpha = \frac{\nu(g)}{2\nu(h)}$. Let $M > 0$ be arbitrary and select t_0 large enough such that $\mathbb{E}[Y_{t_0}^h] \geq 2M$. Then of course we may find a positive constant κ such $\mathbb{E}[Y_{t_0}^h \mathbf{1}_{\{Y_{t_0}^h \leq \kappa\}}] \geq M$. Since Y_t^h and Y_t^g are nondecreasing in t , it follows by (2.45) that

$$\mathbb{P}_x \left(\lim_{t \rightarrow \infty} (Y_t^g - \alpha Y_{t_0}^h) \mathbf{1}_{\{Y_{t_0}^h \leq \kappa\}} \geq 0 \right) = 0.$$

This of course implies, using dominated convergence, that $\liminf_{t \rightarrow \infty} \mathbb{E}[Y_t^g \mathbf{1}_{\{Y_{t_0}^h \leq \kappa\}}] \geq \alpha M$. Since M was arbitrary, this shows that $\mathbb{E}[Y_t^g] \rightarrow \infty$ as $t \rightarrow \infty$, or equivalently that $\int_0^\infty \mathbb{E}_x[g(X_t)] dt = \infty$. Using this property in the proof of Theorem 2.3 we obtain the following corollary.

Corollary 2.4 *For $\lambda^*(f)$ to be strictly monotone at f on the right it is sufficient that there exists some non-trivial Borel measurable function $g: \mathbb{R}^d \rightarrow \mathbb{R}_+$ with compact support satisfying $\lambda^*(f + \varepsilon g) > \lambda^*(f)$ for all $\varepsilon > 0$.*

2.4 Potentials f vanishing at infinity

Let $\mathcal{B}_0(\mathbb{R}^d)$ denote the class of locally bounded Borel measurable functions satisfying $\lim_{R \rightarrow \infty} \sup_{B_R^c} |f| = 0$, and $\mathcal{B}_0^+(\mathbb{R}^d)$ the set of nonnegative functions in $\mathcal{B}_0(\mathbb{R}^d)$ which are not a.e. equal to 0.

We present the following (pinned) multiplicative ergodic theorem (see [29]). Note that the continuity result below on λ^* is stronger than that of [8, Proposition 9.2]. See also Remark 4.1 for continuity of $\lambda^*(f)$ for a larger class of f . Consider the eigenvalue $\lambda''(f)$ defined by

$$\lambda''(f) = \inf \left\{ \lambda : \exists \varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d), \inf_{\mathbb{R}^d} \varphi > 0, \mathcal{L}\varphi + (f - \lambda)\varphi \leq 0 \text{ a.e. in } \mathbb{R}^d \right\}. \quad (2.46)$$

Theorem 2.4 *Let $f \in \mathcal{B}_0(\mathbb{R}^d)$. If the solution of (2.1) is recurrent, then $\lambda^*(f) = \lambda''(f) = \mathcal{E}(f)$. In addition, if the solution of (2.1) is positive recurrent with invariant measure μ , and $\int_{\mathbb{R}^d} f d\mu > 0$, the following hold:*

(a) *for any measurable g with compact support we have*

$$\mathbb{E}_x \left[e^{\int_0^T [f(X_s) - \lambda^*(f)] ds} g(X_T) \right] \xrightarrow{T \rightarrow \infty} C_g \Psi^*(x), \quad (2.47)$$

for some positive constant C_g . Moreover, the corresponding twisted process Y^ is exponentially ergodic.*

(b) *If f_n is a sequence of functions in $\mathcal{B}_0(\mathbb{R}^d)$ satisfying $\sup_n \|f_n\|_\infty < \infty$, and converging to f in $L_{\text{loc}}^1(\mathbb{R}^d)$, and also uniformly outside some compact set $K \subset \mathbb{R}^d$, then $\lambda^*(f_n) \rightarrow \lambda^*(f)$.*

Proof Without loss of generality we may assume that $f > 0$, otherwise we can translate f by a suitable constant. Applying the Itô-Krylov formula to (2.46) it is easy to see that $\mathcal{E}(f) \leq \lambda''(f)$. Also, from [1] we have $\lambda^*(f) \leq \mathcal{E}(f)$. Thus we obtain $\lambda^*(f) \leq \mathcal{E}(f) \leq \lambda''(f)$. If $\lambda^*(f) \geq \lim_{|x| \rightarrow \infty} f(x)$, then by [8, Theorem 1.9 (iii)] we have $\lambda^*(f) = \lambda''(f)$ which in turn implies that $\lambda^*(f) = \mathcal{E}(f) = \lambda''(f)$. Therefore suppose $\lambda^*(f) < \lim_{|x| \rightarrow \infty} f(x)$. Then f is near-monotone, relative to $\lambda^*(f)$, in the sense of [1]. Applying [1, Lemma 2.1] we get $\lambda^*(f) = \mathcal{E}(f) = \lambda''(f)$.

Applying Jensen's inequality it is easy to see that $\mathcal{E}(f) \geq \int f d\mu > 0$. Therefore $\lambda^*(f - f^+) \leq 0 < \lambda^*(f)$. Taking $h = f^+$ and mimicking the arguments of [Theorem 2.1](#) we see that Y^* is exponentially ergodic. Let μ^* be the unique invariant measure of Y^* . Then (2.47) follows from (2.28) and [33, Theorem 1.3.10] with

$$C_g = \int \frac{g}{\Psi^*} d\mu^*.$$

Next we prove part (b). By the first part of the theorem we have $\lambda^*(f_n) = \mathcal{E}(f_n)$ for all n , and by the lower-semicontinuity property of λ^* it holds that $\liminf_{n \rightarrow \infty} \lambda^*(f_n) \geq \lambda^*(f)$. Let $h \in C_c^+(\mathbb{R}^d)$ and $\tilde{f} = f - h$. Then by [Theorem 2.2](#) we have $2\delta := \lambda^*(f) - \lambda^*(\tilde{f}) > 0$. Choose a compact set \mathcal{B} , containing K , such that $\sup_{x \in \mathcal{B}^c} |f_n - f| < \delta$ and $\lambda^*(f_n) > \lambda^*(f) - \delta$ for all large n . Let $(\Psi_n^*, \lambda^*(f_n))$ we denote the principal eigenpair. Then

$$\mathcal{L}\Psi_n^* + f_n \Psi_n^* = \lambda^*(f_n) \Psi_n^*. \quad (2.48)$$

We can choose \mathcal{B} large enough, independent of n , such that

$$\Psi_n^*(x) = \mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f_n(X_t) - \lambda^*(f_n)] dt} \Psi_n^*(X_{\tilde{\tau}}) \right], \quad x \in \mathcal{B}^c, \quad \forall n, \quad (2.49)$$

where $\tilde{\tau} = \tilde{\tau}(\mathcal{B})$. Suppose $\limsup_{n \rightarrow \infty} \lambda^*(f_n) = \Lambda$. It is standard to show that for some positive Ψ , it holds that $\Psi_n^* \rightarrow \Psi$ weakly in $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p > d$, as $n \rightarrow \infty$, and therefore, from (2.48) we have

$$\mathcal{L}\Psi + f\Psi = \Lambda\Psi.$$

Therefore $\Lambda \geq \lambda^*(f)$. Note that on \mathcal{B}^c we have

$$f_n - \lambda^*(f_n) \leq f + \delta - \lambda^*(f_n) \leq f - \lambda^*(f) + 2\delta = f - \lambda^*(\tilde{f})$$

for all n sufficiently large. Since

$$\mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f(X_t) - \lambda^*(\tilde{f})] dt} \right] < \infty,$$

passing to the limit in (2.49), and using the dominated convergence theorem, we obtain that

$$\Psi(x) = \mathbb{E}_x \left[e^{\int_0^{\tilde{\tau}} [f(X_t) - \Lambda] dt} \Psi(X_{\tilde{\tau}}) \right], \quad x \in \mathcal{B}^c.$$

Therefore $\Lambda = \lambda^*(f)$, by [Corollary 2.3](#). This completes the proof. \square

In the rest of the section we show how the previous development can be used to obtain results analogous to those reported in [27], without imposing any regularity assumptions on the coefficients. For $\check{\Psi} = -\log \Psi^* = -\Psi^*$ we have

$$-a^{ij} \partial_{ij} \check{\Psi} - b^i \partial_i \check{\Psi} + \langle \check{\Psi}, a \check{\Psi} \rangle + f = \lambda^*(f). \quad (2.50)$$

Note that (2.50) is a particular form of a more general class of quasilinear pdes of the form

$$-a^{ij} \partial_{ij} \check{\Psi} + H(x, \nabla \check{\Psi}) + f = \lambda^*(f), \quad (2.51)$$

where the function $H(x, p)$, with $(x, p) \in \mathbb{R}^d \times \mathbb{R}^d$, serves as a Hamiltonian. Let f be a non-constant, nonnegative continuous function satisfying $\lim_{|x| \rightarrow \infty} f(x) = 0$. Let $\Lambda_\beta = \lambda^*(\beta f)$. Then by [8, Proposition 2.3 (vii)] we know that $\beta \mapsto \Lambda_\beta$ is non-decreasing, convex. For $a = \text{Id}$, Ichihara studies some qualitative properties of Λ_β in [27] associated to the pde (2.51) and their relation to the recurrence and transience behavior of the process with generator

$$\mathcal{A}^{\check{\Psi}} g = \Delta g - \langle \nabla_p H(x, \nabla \check{\Psi}) \nabla g \rangle, \quad g \in C_c^2(\mathbb{R}^d).$$

It should be noted that for $H(x, p) = -\langle b(x), p \rangle + \langle p, a(x)p \rangle$, $\mathcal{A}^{\check{\Psi}}$ is the generator of the twisted process Y^* corresponding to λ^* . One of the key assumptions in [27, Assumption (H1) (i)] is that $H(x, p) \geq H(x, 0) = 0$ for all x, p . Note that this forces b to be 0.

Let $\beta_c := \inf \{ \beta \in \mathbb{R} : \Lambda_\beta > \lim_{\beta \rightarrow -\infty} \Lambda_\beta \}$. It is easy to see that $\beta_c \in [-\infty, \infty]$. The following result is an extension of [27, Theorems 2.2 and 2.3] to measurable drifts b and potentials f .

Theorem 2.5 *Let $f \in \mathcal{B}_0^+(\mathbb{R}^d)$. Then the twisted process $Y^* = Y^*(\beta)$ corresponding to the eigenpair $(\Psi_\beta^*, \Lambda_\beta)$ is transient for $\beta < \beta_c$, exponentially ergodic for $\beta > \beta_c$, and, provided $f = 0$ a.e. outside some compact set, it is recurrent for $\beta = \beta_c$. In addition, the following hold.*

- (i) *If \mathcal{L}^0 is self-adjoint (i.e., $\mathcal{L}^0 = \partial_i(a^{ij}\partial_j)$) with a bounded, uniformly elliptic and radially symmetric in \mathbb{R}^d , and the solution of (2.1) is transient, then $\beta_c \geq 0$. Also $\Lambda_\beta \geq 0$ for all $\beta \in \mathbb{R}$.*
- (ii) *Provided that the solution of (2.1) is recurrent, then $\beta_c < 0$ if it is exponentially ergodic, and $\beta_c = 0$ otherwise.*
- (iii) *Assume that $\beta > \beta_c$, and that (2.1) is recurrent in the case that $\Lambda_\beta \leq 0$. Let Ψ_β^* and μ_β^* denote the ground state and the invariant probability measure of the ground state diffusion, respectively, corresponding to Λ_β . Then it holds that*

$$\Lambda_\beta = \mu_\beta^*(\beta f - \langle \nabla \Psi_\beta^*, a \nabla \Psi_\beta^* \rangle), \quad (2.52)$$

where, as usual, $\Psi_\beta^* = \log \Psi_\beta^*$.

Proof The first part of the proof follows from Theorems 2.1 and 2.3 and Corollary 2.4. Next we proceed to prove (i). Suppose $\beta_c < 0$. Then $Y^* = Y^*(0)$, the twisted process corresponding to Λ_0 is exponentially ergodic. By [8, Theorem 1.9(i)-(ii)] we have $\Lambda_0 = \mathcal{E}(0) = 0$. Moreover, $\Psi_0^* = 1$ is a principal eigenfunction. Therefore the twisted process is given by (2.1) which is transient by assertion. This is a contradiction. Hence $\beta_c \geq 0$. Now $\beta \mapsto \Lambda_\beta$ being convex this implies that Λ_β is constant in $(-\infty, \beta_c] \ni 0$. Hence $\Lambda_\beta = \Lambda_0 = 0$ for $\beta \leq \beta_c$. This proves (i).

We now turn to part (ii). By Theorem 2.4 we have $\Lambda_\beta = \mathcal{E}(\beta f)$. We first show that if the solution of (2.1) is recurrent then $\lambda^*(\beta f) > 0$, whenever $\beta > 0$. Indeed, arguing by contradiction, if $\lambda^*(\beta f) = 0$ for some $\beta > 0$, then $\mathcal{L}\Psi_\beta^* = -\beta f \Psi_\beta^*$ on \mathbb{R}^d , which implies that $\Psi_\beta^*(X_t)$ is a nonnegative supermartingale and since it is integrable, it converges a.s. Since the process is recurrent, this implies that Ψ_β^* must equal to a constant, which, in turn, necessitates that $f = 0$, a contradiction. This proves the claim and this implies that if the solution of (2.1) is recurrent then $\beta_c \leq 0$. Now suppose that β_c is negative. Then the twisted process corresponding to $\beta = 0$ is exponentially ergodic by Theorem 2.1. Since $\Psi_0^* = 1$ is a principal eigenfunction for $\beta = 0$, the ground state diffusion agrees with (2.1), which implies that the latter is exponentially ergodic.

Now suppose that X , and therefore also $Y^*(0)$ is exponentially ergodic. It then follows from Theorem 2.2 that $\beta \mapsto \Lambda_\beta$ is strictly monotone at 0. This of course implies that $\beta_c < 0$. The proof of part (ii) is now complete.

Next we prove part (iii). We distinguish two cases.

Case 1. Suppose $\Lambda_\beta > 0$.

Let $\tilde{\Psi} = \tilde{\Psi}_\beta = (\Psi_\beta^*)^{-1}$ and $\tilde{\psi} := \log \tilde{\Psi}$. Then $\tilde{\Psi}$ satisfies

$$\tilde{\mathcal{L}}^{\Psi_\beta^*} \tilde{\Psi} = (\beta f - \Lambda_\beta) \tilde{\Psi} \quad (2.53)$$

Since $\beta f \in \mathcal{B}_0(\mathbb{R}^d)$, there exists $\varepsilon_0 > 0$ and a ball \mathcal{B} such that $\beta f - \Lambda_\beta < -\varepsilon_0$ for all $x \in \mathcal{B}^c$. Applying the Feynman–Kac formula it follows from [1, Lemma 2.1] that $\inf_{\mathbb{R}^d} \tilde{\Psi} = \min_{\bar{\mathcal{B}}} \tilde{\Psi}$. Thus $\tilde{\Psi}$ is bounded away from 0 on \mathbb{R}^d . Let Y^* denote the ground state process corresponding to the eigenvalue Λ_β . Simplifying the notation we let $\tilde{\mathbb{E}}_x^* := \tilde{\mathbb{E}}_x^{\Psi_\beta^*}$. By the exponential Foster–Lyapunov equation (2.53) we have that (see [2, Lemma 2.5.5])

$$\tilde{\mathbb{E}}_x^*[\tilde{\Psi}(Y_t^*)] \leq C_0 + \tilde{\Psi}(x) e^{-\varepsilon_0 t} \quad \forall t \geq 0. \quad (2.54)$$

Using this estimate together with $\inf_{\mathbb{R}^d} \tilde{\Psi} > 0$ on $\tilde{\psi} = \log \tilde{\Psi}$ we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \tilde{\mathbb{E}}_x^*[\tilde{\psi}(Y_t^*)] = 0. \quad (2.55)$$

Next we show that

$$\lim_{R \rightarrow \infty} \tilde{\mathbb{E}}_x^*[\tilde{\psi}(Y_{t \wedge \tau_R}^*)] = \tilde{\mathbb{E}}_x^*[\tilde{\psi}(Y_t^*)], \quad (2.56)$$

where τ_R denotes the exit time from the ball B_R . First, there exists some constant k_0 such that $(\beta f - \Lambda_\beta)\check{\Psi} \leq k_0$ on \mathbb{R}^d . Thus $\tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_t^*)] \leq k_0 t + \check{\Psi}(x)$ by (2.53), and of course also $\tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{t \wedge \tau_R}^*)] \leq k_0 t + \check{\Psi}(x)$. Let $\Gamma(R, m) := \{x \in \partial B_R : |\check{\Psi}(x)| \geq m\}$ for $m \geq 1$. Then

$$\begin{aligned} \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{\tau_R}^*) \mathbf{1}_{\{t \geq \tau_R\}}] &\leq m \tilde{\mathbb{P}}_x^*(t \geq \tau_R) + \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{\tau_R}^*) \mathbf{1}_{\Gamma(R, m)}(Y_{\tau_R}^*) \mathbf{1}_{\{t \geq \tau_R\}}] \\ &\leq m \tilde{\mathbb{P}}_x^*(t \geq \tau_R) + (k_0 t + \check{\Psi}(x)) \sup_{\Gamma(R, m)} \frac{\check{\Psi}}{\check{\Psi}} \\ &\leq m \tilde{\mathbb{P}}_x^*(t \geq \tau_R) + \frac{m}{e^m} (k_0 t + \check{\Psi}(x)). \end{aligned}$$

Taking limits as $R \rightarrow \infty$, and since $m \in \mathbb{R}_+$ is arbitrary, it follows that

$$\lim_{R \rightarrow \infty} \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{\tau_R}^*) \mathbf{1}_{\{t \geq \tau_R\}}] = 0. \quad (2.57)$$

Write

$$\tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{t \wedge \tau_R}^*)] = \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_t^*) \mathbf{1}_{\{t < \tau_R\}}] + \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{\tau_R}^*) \mathbf{1}_{\{t \geq \tau_R\}}]. \quad (2.58)$$

Without loss of generality we assume $\check{\Psi} \geq 1$. Since $|\check{\Psi}| \leq \check{\Psi}$, Fatou's lemma gives

$$\begin{aligned} \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_t^*)] &\leq \liminf_{R \rightarrow \infty} \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_t^*) \mathbf{1}_{\{t < \tau_R\}}] \\ &\leq \limsup_{R \rightarrow \infty} \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_t^*) \mathbf{1}_{\{t < \tau_R\}}] \leq \tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_t^*)], \end{aligned}$$

and use this together with (2.57) and (2.58) to obtain (2.56).

We write (2.50) as

$$\begin{aligned} 0 &= a^{ij} \partial_{ij} \check{\Psi} + b^i \partial_i \check{\Psi} + \min_{u \in \mathbb{R}^d} [2 \langle a u, \nabla \check{\Psi} \rangle + \langle u, a u \rangle] - \beta f + \Lambda_\beta \\ &= a^{ij} \partial_{ij} \check{\Psi} + b^i \partial_i \check{\Psi} - 2 \langle \nabla \check{\Psi}, a \nabla \check{\Psi} \rangle + \langle \nabla \check{\Psi}, a \nabla \check{\Psi} \rangle - \beta f + \Lambda_\beta \\ &= \tilde{\mathcal{L}}^{\Psi^*} \check{\Psi} + \langle \nabla \check{\Psi}, a \nabla \check{\Psi} \rangle - \beta f + \Lambda_\beta \end{aligned} \quad (2.59)$$

Let $F := \langle \nabla \check{\Psi}, a \nabla \check{\Psi} \rangle - \beta f$. Applying the Itô–Krylov formula to (2.59), we obtain

$$\tilde{\mathbb{E}}_x^*[\check{\Psi}(Y_{t \wedge \tau_R}^*)] - \check{\Psi}(x) + \tilde{\mathbb{E}}_x^* \left[\int_0^{t \wedge \tau_R} F(Y_s^*) ds \right] + \Lambda_\beta \tilde{\mathbb{E}}_x^*[t \wedge \tau_R] = 0. \quad (2.60)$$

Letting $R \rightarrow \infty$ in (2.60), using (2.56), then dividing by t and letting $t \rightarrow \infty$, using (2.55) and Birkhoff's ergodic theorem, we obtain

$$\mu_\beta^*(\langle \nabla \check{\Psi}, a \nabla \check{\Psi} \rangle - \beta f) + \Lambda_\beta = 0,$$

which is the assertion in part (iii).

Case 2. Suppose $\Lambda_\beta \leq 0$ and (2.1) is recurrent. The case $\Lambda_\beta = 0$ is then trivial, since $\nabla \Psi_0^* = 0$, so we assume that $\Lambda_\beta < 0$.

Then (2.1) is exponentially ergodic by part (ii), and thus Ψ_β^* is bounded below in \mathbb{R}^d by [1, Lemma 2.1]. With $\psi^* = \Psi_\beta^* = \log \Psi_\beta^*$, in analogy to (2.59) we have

$$\tilde{\mathcal{L}}^{\Psi^*} \psi^* - \langle \nabla \psi^*, a \nabla \psi^* \rangle + \beta f - \Lambda_\beta = 0. \quad (2.61)$$

We claim that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \tilde{\mathbb{E}}_x^*[\psi^*(Y_t^*)] = 0, \quad \text{and} \quad \lim_{R \rightarrow \infty} \tilde{\mathbb{E}}_x^*[\psi^*(Y_{t \wedge \tau_R}^*)] = \tilde{\mathbb{E}}_x^*[\psi^*(Y_t^*)] \quad (2.62)$$

where as defined earlier, $\widetilde{\mathbb{E}}_x^* = \widetilde{\mathbb{E}}_x^{\Psi^*}$, and Y^* denotes the ground state process. Assuming (2.62) is true, we first apply the Itô–Krylov formula to (2.61) to obtain the analogous equation to (2.60), and then take limits and use Birkhoff’s ergodic theorem to establish (2.52).

It remains to prove (2.62). Choose $\varepsilon > 0$ so that $\beta > \beta - \varepsilon > \beta_c$, and let $\Psi_{\beta-\varepsilon}^*$ denote the ground state corresponding to $\Lambda_{\beta-\varepsilon}$. We choose a ball \mathcal{B} such that

$$\varepsilon f(x) < \frac{1}{2}(\Lambda_\beta - \Lambda_{\beta-\varepsilon}) \quad \forall x \in \mathcal{B}^c. \quad (2.63)$$

Since f vanishes at infinity, and $\Lambda_\beta > \Lambda_{\beta-\varepsilon}$, there exists a constant $\alpha > 1$ and a ball also denoted as \mathcal{B} , such that

$$\alpha(\beta f(x) - \Lambda_\beta) < (\beta - \varepsilon)f(x) - \Lambda_{\beta-\varepsilon} \quad \forall x \in \mathcal{B}^c. \quad (2.64)$$

Since the ground state processes corresponding to the principal eigenvalues Λ_β and $\Lambda_{\beta-\varepsilon}$ are ergodic we have from Lemma 2.7 that

$$\begin{aligned} \Psi_\beta^*(x) &= \mathbb{E}_x \left[e^{\int_0^\tau [\beta f(X_s) - \Lambda_\beta] ds} \Psi_\beta^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \\ \Psi_{\beta-\varepsilon}^*(x) &= \mathbb{E}_x \left[e^{\int_0^\tau [(\beta-\varepsilon)f(X_s) - \Lambda_{\beta-\varepsilon}] ds} \Psi_{\beta-\varepsilon}^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right], \end{aligned} \quad (2.65)$$

for all $x \in \mathcal{B}^c$ where $\tau = \tau(\mathcal{B}^c)$. By (2.27), the function $\widetilde{\Psi}_\varepsilon := \frac{\Psi_{\beta-\varepsilon}^*}{\Psi_\beta^*}$ satisfies

$$\widetilde{\mathcal{L}}^{\Psi^*} \widetilde{\Psi}_\varepsilon = (\Lambda_{\beta-\varepsilon} - \Lambda_\beta + \varepsilon f) \widetilde{\Psi}_\varepsilon. \quad (2.66)$$

Applying the Feynman–Kac formula to (2.66), using (2.63), it follows as in [1, Lemma 2.1] that $\inf_{\mathbb{R}^d} \widetilde{\Psi}_\varepsilon = \min_{\mathcal{B}} \widetilde{\Psi}_\varepsilon$. Thus $\widetilde{\Psi}_\varepsilon$ is bounded away from 0 on \mathbb{R}^d .

Let $\kappa := \min_{\mathcal{B}} \frac{\Psi_{\beta-\varepsilon}^*}{\Psi_\beta^*}$. Then by (2.64) and (2.65) we obtain

$$\begin{aligned} \widetilde{\Psi}_\varepsilon(x) &\geq \frac{\kappa}{\Psi_\beta^*(x)} \mathbb{E}_x \left[e^{\int_0^\tau \alpha[\beta f(X_s) - \Lambda_\beta] ds} (\Psi_\beta^*(X_\tau))^\alpha \mathbf{1}_{\{\tau < \infty\}} \right] \\ &\geq \frac{\kappa}{\Psi_\beta^*(x)} \left(\mathbb{E}_x \left[e^{\int_0^\tau [\beta f(X_s) - \Lambda_\beta] ds} \Psi_\beta^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] \right)^\alpha \\ &\geq \kappa (\Psi_\beta^*(x))^{\alpha-1} \quad \forall x \in \mathcal{B}^c. \end{aligned}$$

Therefore, for some constant κ_1 we have

$$\psi^* \leq \kappa_1 + \frac{1}{\alpha-1} \log \widetilde{\Psi}_\varepsilon \quad \text{on } \mathbb{R}^d. \quad (2.67)$$

Let $\varepsilon_0 := \frac{1}{2}(\Lambda_\beta - \Lambda_{\beta-\varepsilon})$. From (2.63) and exponential Foster–Lyapunov equation (2.66) we deduce that (2.54) holds for $\widetilde{\Psi}_\varepsilon$. Thus the first equation in (2.62) follows directly from (2.54) and (2.67) and the fact $\inf_{\mathbb{R}^d} \psi^* > -\infty$, while the second one follows by repeating the argument leading to (2.57). This completes the proof. \square

Remark 2.1 The assumption that (2.1) is recurrent in the case that $\Lambda_\beta < 0$ in Theorem 2.5 (iii) is equivalent to the statement that $\lambda^*(0) = 0$. Note that as shown in [28, Theorem 2.1], unless $\lambda^*(0) = 0$, then (2.52) does not hold if $\Lambda_\beta < 0$.

If (2.1) is not recurrent, then it is possible that $\beta_c < 0$ and also that $\Lambda_\beta < 0$ for $\beta \geq 0$. Consider a diffusion with $d = 1$, $b(x) = 2x$, and $\sigma(x) = \sqrt{2}$. Then, we have $\mathcal{L}\varphi = -\varphi$ for $\varphi(x) = \frac{1}{2}e^{-x^2}$. Thus $\hat{\Lambda}_0 \leq -1$, where $\hat{\Lambda}_0$ denotes the eigenvalue in (2.5) for $f = 0$. Thus by Lemma 2.2 (b) $\lambda^*(0) \leq -1$. Since the twisted process corresponding to φ is exponentially ergodic, we must have $\lambda^*(0) = -1$ by Theorem 2.1 (c), and thus φ is the ground state. Theorem 2.1 (b) then asserts that $\beta \mapsto \Lambda_\beta$ is strictly increasing at $\beta = 0$. Therefore $\beta_c < 0$. Observe that the ground state diffusion is an Ornstein–Uhlenbeck process having a Gaussian stationary distribution of mean 0 and variance $1/2$. An easy computation reveals that $\mu^*(-\langle \nabla \psi^*, a \nabla \psi^* \rangle) = -2$ which is smaller than $\lambda^*(0)$.

The conclusion of [Theorem 2.5](#) (iii) can be sharpened. Consider the controlled diffusion

$$dZ_t = (b(Z_t) + 2a(Z_t)v(Z_t)) dt + \sigma(Z_t) dW_t. \quad (2.68)$$

Here $v: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a locally bounded Borel measurable map. Let $\widehat{\mathcal{U}}_{\text{SM}}$ denote the class of such maps. These are identified with the class of locally bounded stationary Markov controls. Let $\widehat{\mathcal{U}}_{\text{SSM}} \subset \widehat{\mathcal{U}}_{\text{SM}}$ be the collection of those v under which the diffusion in (2.68) is ergodic, and denote by $\widehat{\mu}_v$ the associated invariant probability measure. We let $\mathcal{A}_v := \mathcal{L} + 2av \cdot \nabla$, and use the symbol $\widehat{\mathbb{E}}_x^v$ to denote the expectation operator associated with (2.68).

In order to simplify the notation, we use the norm $\|v\|_a := \sqrt{\langle v, av \rangle}$. For $v \in \widehat{\mathcal{U}}_{\text{SM}}$ we define

$$F_v(z) := \|v(z)\|_{a(z)}^2 - \beta f(z),$$

$$\mathcal{J}_x(v) := \limsup_{T \rightarrow \infty} \frac{1}{T} \widehat{\mathbb{E}}_x^v \left[\int_0^T F_v(Z_s) ds \right],$$

and $\overline{\mathcal{J}}_x = \inf_{v \in \widehat{\mathcal{U}}_{\text{SM}}} \mathcal{J}_x(v)$.

Theorem 2.6 *Assume that $f \in \mathcal{B}_0^+(\mathbb{R}^d)$ and $\beta > \beta_c$. Then the following hold*

(a) *If $\Lambda_\beta > 0$, then we have*

$$\overline{\mathcal{J}}_x = \mathcal{J}_x(\nabla \psi^*) = -\Lambda_\beta \quad \forall x \in \mathbb{R}^d. \quad (2.69)$$

In addition, if $v \in \widehat{\mathcal{U}}_{\text{SM}}$ satisfies $\mathcal{J}_x(v) = \overline{\mathcal{J}}_x$, then $v = \nabla \psi_\beta^$ a.e.*

(b) *If $\Lambda_\beta \leq 0$ and (2.1) is recurrent then (2.69) holds, and $v = \nabla \psi_\beta^*$ is the a.e. unique control in $\widehat{\mathcal{U}}_{\text{SSM}}$ which satisfies $\mathcal{J}_x(v) = \overline{\mathcal{J}}_x$.*

(c) *If $\Lambda_\beta < 0$ and (2.1) is not recurrent, then $\overline{\mathcal{J}}_x = 0$ for all $x \in \mathbb{R}^d$.*

Proof We start with part (a). By [Theorem 2.5](#) (iii), we have $\mathcal{J}_x(\nabla \psi_\beta^*) = -\Lambda_\beta$ in both of cases (a) and (b). It suffices then to show that if $\mathcal{J}_x(v) \leq -\Lambda_\beta$ for some $v \in \widehat{\mathcal{U}}_{\text{SM}}$ then $v = \nabla \psi_\beta^*$ a.e. in \mathbb{R}^d . Let such a control v be given. Then (2.68) must be positive recurrent under v , for otherwise we must have $\mathcal{J}_x(v) \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \widehat{\mathbb{E}}_x^v [\int_0^T -\beta f(Z_s) ds] \geq 0$. Therefore

$$\mathcal{J}_x(v) = \int_{\mathbb{R}^d} F_v(z) \widehat{\mu}_v(dz) \leq -\Lambda_\beta < 0,$$

where $\widehat{\mu}_v$, as defined earlier, denotes the invariant probability measure associated with $\mathcal{A}_v := \mathcal{L} + 2\langle v, a \nabla \rangle$. Since v is locally bounded, $\liminf_{|z| \rightarrow \infty} F_v(z) > \mathcal{J}_x(v)$, and F_v is integrable with respect to $\widehat{\mu}_v$, we can assert the existence of a solution $\check{\phi} \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ to the Poisson equation

$$\mathcal{L}\check{\phi}(x) + 2a(x)v(x) \cdot \nabla \check{\phi}(x) + F_v(x) = \mathcal{J}_x(v), \quad (2.70)$$

which is bounded below in \mathbb{R}^d (see Lemma 3.7.8 (d) in [2]). It follows by (2.70) that $\Phi := e^\phi$, $\phi = -\check{\phi}$, satisfies

$$\mathcal{L}\Phi + (\beta f - \|v - \nabla \phi\|_a^2) \Phi = -\mathcal{J}_x(v) \Phi. \quad (2.71)$$

This shows that $(\Phi, -\mathcal{J}_x(v))$ is an eigenpair for $\mathcal{L}^{\check{F}}$, with $\check{F} := \beta f - \|v - \nabla \phi\|_a^2$. The twisted process with generator $\tilde{\mathcal{L}} = \mathcal{L} + 2a\nabla \phi \cdot \nabla$ satisfies

$$\tilde{\mathcal{L}}\check{\phi} + \|\nabla \phi\|_a^2 + \|v - \nabla \phi\|_a^2 - \beta f = \mathcal{J}_x(v). \quad (2.72)$$

Since $\check{\phi}$ is bounded below in \mathbb{R}^d and $\mathcal{J}_x(v) < 0$, (2.72) shows that the twisted process is positive recurrent. We claim that $-\mathcal{J}_x(v)$ is the principal eigenvalue of $\mathcal{L}^{\check{F}}$. Indeed, if $\lambda^*(\check{F}) < -\mathcal{J}_x(v)$ then by the proof of [Lemma 2.3](#) and for any $g \in C_c^+(\mathbb{R}^d)$ we obtain, for all n large,

$$\Phi(x) \widehat{\mathbb{E}}_x^\phi [g(Y_T) \Phi^{-1}(Y_T) \mathbf{1}_{\{T \leq \tau_n\}}] = \Psi^*(x) \widehat{\mathbb{E}}_x^{\Psi^*} [e^{[\lambda^*(\check{F}) + \mathcal{J}_x(v)]T} g(Y_T^*) (\Psi^*)^{-1}(Y_T^*) \mathbf{1}_{\{T \leq \tau_n\}}]$$

$$\leq \Psi^*(x) \left(\sup_{\mathbb{R}^d} \frac{g}{\Psi^*} \right) e^{[\lambda^*(\tilde{F}) + \mathcal{J}_x(v)]T},$$

where τ_n denotes the first exit time from B_n . By first letting $n \rightarrow \infty$, and then integrating with respect to T we obtain

$$\int_0^\infty \tilde{\mathbb{E}}^\varphi [g(Y_t) \Phi^{-1}(Y_t)] dt < \infty.$$

But this contradicts the positive recurrence of the twisted process corresponding to $\tilde{\mathcal{L}}$. Therefore $-\mathcal{J}_x(v)$ must be the principal eigenvalue of $\mathcal{L}^{\tilde{F}}$, which implies that

$$\begin{aligned} -\mathcal{J}_x(v) &= \lambda^*(\beta f - \|v - \nabla \phi\|_a^2) \\ &\leq \Lambda_\beta. \end{aligned} \quad (2.73)$$

Thus we have shown that $\mathcal{J}_x(v) = -\Lambda_\beta$. The strict monotonicity of Λ_β at βf together with (2.73) imply that $v = \nabla \phi$ a.e. in \mathbb{R}^d . In turn, (2.71) and the uniqueness of the ground state imply that $\Phi = \Psi_\beta^*$, up to a multiplication by a positive constant, and therefore we have shown that $v = \nabla \Psi_\beta^*$ a.e. in \mathbb{R}^d , thus completing the proof of part (a).

We continue with part (b). The case $\Lambda_\beta = 0$ is trivial, so assume that $\Lambda_\beta < 0$. Then of course (2.1) is exponentially ergodic by Theorem 2.5(ii). Since $\Lambda_0 = 0$, by Theorem 2.4, and $\beta \mapsto \Lambda_\beta$ is strictly increasing in (β_c, ∞) , we have $\beta < 0$. Thus Ψ_β^* is bounded away from 0 in \mathbb{R}^d by [1, Lemma 2.1]. Let $v \in \hat{\mathcal{U}}_{\text{SM}}$, and $\tilde{\Psi} = -\Psi_\beta^*$. We have

$$\mathcal{L}\tilde{\Psi} + 2\langle v, a\nabla\tilde{\Psi} \rangle - \|v + \nabla\tilde{\Psi}\|_a^2 + F_v = -\Lambda_\beta. \quad (2.74)$$

Since $\tilde{\Psi}$ is bounded above in \mathbb{R}^d , it follows from (2.74) in a standard manner that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \hat{\mathbb{E}}_x^v \left[\int_0^T F_v(Z_s) ds \right] \geq -\Lambda_\beta.$$

We next show uniqueness in $\hat{\mathcal{U}}_{\text{SSM}}$ of the optimal control $\nabla \Psi_\beta^*$. Let $v \in \hat{\mathcal{U}}_{\text{SSM}}$ and suppose $\mathcal{J}_x(v) = -\Lambda_\beta$. By the Itô–Krylov formula and Fatou’s lemma and since $\tilde{\Psi}$ is bounded above, we obtain from (2.74) that

$$\hat{\mathbb{E}}_x^v [\tilde{\Psi}(Z_t)] - \tilde{\Psi}(x) - \hat{\mathbb{E}}_x^v \left[\int_0^t G_v(Z_s) ds \right] + \hat{\mathbb{E}}_x^v \left[\int_0^t F_v(Z_s) ds \right] \geq -t\Lambda_\beta, \quad (2.75)$$

with $G_v(z) := \langle (v(z) + \nabla \tilde{\Psi}(z)), a(z)(v(z) + \nabla \tilde{\Psi}(z)) \rangle$. Dividing (2.75) by t and taking limits as $t \rightarrow \infty$, we obtain $-\hat{\mu}_v(G_v) + \mathcal{J}_x(v) \geq -\Lambda_\beta$, which implies that $\hat{\mu}_v(G_v) = 0$, since G_v is nonnegative. Thus $G_v = 0$ a.e. in \mathbb{R}^d , which implies that $v = -\nabla \tilde{\Psi} = \nabla \Psi_\beta^*$ a.e. in \mathbb{R}^d .

We now turn to part (c). It is evident that under the control $v = 0$, since the diffusion in (2.68) is transient and f vanishes at infinity, we have $\lim_{t \rightarrow \infty} \frac{1}{t} \hat{\mathbb{E}}_x^v [F_0(Z_t)] = 0$. It is also clear that under any control $v \in \hat{\mathcal{U}}_{\text{SM}} \setminus \hat{\mathcal{U}}_{\text{SSM}}$ we have $\lim_{t \rightarrow \infty} \frac{1}{t} \hat{\mathbb{E}}_x^v [F_0(Z_t)] \geq 0$. Suppose that under some $v \in \hat{\mathcal{U}}_{\text{SSM}}$, we have $\liminf_{t \rightarrow \infty} \frac{1}{t} \hat{\mathbb{E}}_x^v [F_v(Z_t)] = \mathcal{J}_x(v) < 0$. Then there exists a solution $\check{\phi}$ to the Poisson equation (2.70) which is bounded below in \mathbb{R}^d . Thus following the proof of Case 1 in part (a) we obtain by (2.73) that $\mathcal{J}_x(v) \geq -\Lambda_\beta$ which is a contradiction. We have therefore shown that $\mathcal{J}_x(v) \geq 0$ for all $v \in \hat{\mathcal{U}}_{\text{SM}}$, which implies that 0 is the optimal value in the class of controls $\hat{\mathcal{U}}_{\text{SM}}$. \square

Remark 2.2 The assumption that f is nonnegative can be relaxed to $f \in \mathcal{B}_0(\mathbb{R}^d)$. From the proof of Theorem 2.2 we note that if $\lambda^*(f+h) < \lambda^*(f)$ for some $h \in \mathcal{B}_0(\mathbb{R}^d)$, then the ground state diffusion corresponding to $\lambda^*(f)$ is geometrically ergodic. Moreover, due to [8, Proposition 2.3(vii)] the function $\beta \mapsto \lambda^*(\beta f)$ is convex for every $f \in \mathcal{B}_0(\mathbb{R}^d)$. Instead of a critical value β_c , we can define a critical value λ_c by $\lambda_c := \inf_{\beta \in \mathbb{R}} \Lambda_\beta$. Then if we replace the condition $\beta > \beta_c$ by $\Lambda_\beta > \lambda_c$ as done in [28], it is evident that $\lambda^*(\beta f)$ is strictly monotone at βf and the results in Theorem 2.5 (iii) and Theorem 2.6 still hold, provided $\Lambda_\beta \neq 0$, and the proofs are the same.

The results in [Theorem 2.6](#) (b) can be also stated for nonstationary controls. Consider the controlled diffusion

$$dZ_t = (b(Z_t) + 2a(Z_t)U_t) dt + \sigma(Z_t) dW_t. \quad (2.76)$$

Here $U = \{U_t\}$ is an \mathbb{R}^d -valued control process which is jointly measurable in $(t, \omega) \in [0, \infty) \times \bar{\Omega}$, and are *nonanticipative*: for $t > s$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s := \text{the completion of } \cap_{y>s} \sigma(X_0, W_r, U_r : r \leq y) \text{ relative to } (\mathfrak{F}, \mathbb{P}).$$

Let $f \in \mathcal{B}_0(\mathbb{R}^d)$, not necessarily nonnegative. Assume that $\Lambda_\beta > \lambda_c$, $\Lambda_\beta \leq 0$ and [\(2.1\)](#) is recurrent (see [Remark 2.2](#)). Suppose that under U , the diffusion in [\(2.76\)](#) has a unique weak solution. Then

$$\mathcal{J}_x(U) := \limsup_{T \rightarrow \infty} \frac{1}{T} \widehat{\mathbb{E}}_x^U \left[\int_0^T [\langle U_s, a(Z_s)U_s \rangle - \beta f(Z_s)] ds \right] \geq -\Lambda_\beta.$$

We can prove this as follows. By [\(2.59\)](#) we obtain

$$\mathcal{L}\check{\Psi}(z) + 2\langle u, a\nabla\check{\Psi} \rangle + F_u(z) \geq -\Lambda_\beta$$

for all $u, z \in \mathbb{R}^d$, and we apply the Itô–Krylov formula and Fatou’s lemma (using the fact that $\check{\Psi}$ is bounded above) with $u = U_t$ to obtain analogously to [\(2.75\)](#) that

$$\widehat{\mathbb{E}}_x^U [\check{\Psi}(Z_t)] - \check{\Psi}(x) + \widehat{\mathbb{E}}_x^U \left[\int_0^t F_{U_s}(Z_s) ds \right] \geq -t\Lambda_\beta. \quad (2.77)$$

Dividing [\(2.77\)](#) by t and letting $t \rightarrow \infty$, we obtain

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \widehat{\mathbb{E}}_x^U \left[\int_0^t F_{U_s}(Z_s) ds \right] \geq -\Lambda_\beta.$$

2.4.1 Strong duality

The optimality result in [Theorem 2.6](#) can be strengthened. Consider the class of *infinitesimal ergodic occupation measures*, i.e., measures $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ which satisfy

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{A}_u g(x) \pi(dx, du) = 0 \quad \forall g \in C_c^\infty(\mathbb{R}^d), \quad (2.78)$$

with $\mathcal{A}_u := \mathcal{L} + \langle 2au, \nabla \rangle$. Disintegrate these as

$$\pi(dx, du) = \eta_v(dx) \nu(du|x),$$

and denote this as $\pi = \eta_v \otimes \nu$. Since $\int |u|^2 \eta_v(dx) \nu(du|x) \geq \int |\hat{v}(x)|^2 \eta_v(dx)$ where $\hat{v}(x) = \int v(du|x)$, and since $\eta_v(dx) \delta_{\hat{v}(x)}(du)$ is also an ergodic occupation measure, it is enough to consider the class of infinitesimal ergodic occupation measures π that correspond to a precise control v , i.e., a Borel measurable map from \mathbb{R}^d to \mathbb{R}^d . We denote this class by \mathcal{M} . Thus for $\pi = \eta_v \otimes \nu \in \mathcal{M}$, [\(2.78\)](#) takes the form $\int_{\mathbb{R}^d} \mathcal{A}_v g(x) \eta_v(dx) = 0$. Note though that v is not necessarily locally bounded, so this class of controls is larger than $\hat{\mathcal{U}}_{\text{SSM}}$.

In [Theorem 2.7](#) below we use the fact that if η_v has density $\rho_v \in L^q(\mathbb{R}^d)$, for $q > 1$, and $v \in L^s(\mathbb{R}^d; \eta_v)$, $s \geq 1$, then

$$\int_{\mathbb{R}^d} \mathcal{A}_v g(x) \eta_v(dx) = 0$$

for any $g \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, all $p > d$, and with compact support. This can be seen as follows. Apply a smooth mollifier family $\{\chi_r, r > 0\}$ with compact support to g , so that [\(2.78\)](#) can be applied to the mollified function $g * \chi_r$, where ‘ $*$ ’ denotes convolution. Then separate terms, and applying the Hölder inequality on $|\int (\mathcal{L}g - \mathcal{L}(g * \chi_r)) \rho_v|$, show that this term tends to 0 as $r \searrow 0$, using the convergence of $\mathcal{L}(g * \chi_r)$ to $\mathcal{L}g$ in $L^p(\mathbb{R}^d)$. Similarly apply the Hölder inequality to

$|\int \langle 2av, \nabla(g - g * \chi_r) \rangle \rho_v|$ to extract a term of the form $\int |2av|^s \rho_v$, which is bounded, and use the technique mentioned above on the remaining one.

We also use the following property. If the drift has at most affine growth, then it is straightforward to show that for any open ball \mathcal{B} the map $x \mapsto \mathbb{E}_x[\tau(\mathcal{B}^c)]$ is inf-compact. This together with the stochastic representation in (2.17), and Jensen's inequality, show that if $\Lambda_\beta > \lambda_c$ (see Remark 2.2) then Ψ_β^* is inf-compact when $\Lambda_\beta < 0$ and (2.1) is recurrent, and $\tilde{\Psi} = (\Psi_\beta^*)^{-1}$ is inf-compact when $\Lambda_\beta > 0$.

The theorem that follows shows that there is no optimality gap between the primal problem which consists of minimizing $\int F_u(x) \pi(dx, du)$ subject to the constraint (2.78), and the dual problem which amounts to a maximization over subsolutions of the HJB equation, as described in section 1. This theorem is stated for $f \in \mathcal{B}_0(\mathbb{R}^d)$ which is not necessarily nonnegative as discussed in Remark 2.2.

Theorem 2.7 Assume that $f \in \mathcal{B}_0(\mathbb{R}^d)$ and

- (i) $\Lambda_\beta > \lambda_c$ (see Remark 2.2), and either $\Lambda_\beta > 0$, or $\Lambda_\beta \leq 0$ and (2.1) is recurrent,
- (ii) The coefficients a and b are bounded, and a is uniformly strictly elliptic.

Then any $\pi = \eta_v \otimes v \in \mathcal{M}$ such that $\int_{\mathbb{R}^d} F_v d\eta_v < \infty$ satisfies

$$\int_{\mathbb{R}^d} F_v d\eta_v = -\Lambda_\beta + \int_{\mathbb{R}^d} \|v - \nabla \Psi_\beta^*\|_a^2 d\eta_v. \quad (2.79)$$

In addition, if $\pi = \eta_v \otimes v \in \mathcal{M}$ is optimal, i.e., if it satisfies $\int_{\mathbb{R}^d} F_v d\eta_v = -\Lambda_\beta$, then $v = \nabla \Psi_\beta^*$ a.e. in \mathbb{R}^d and $\eta_v = \mu_\beta^*$.

Proof We first consider the case $\Lambda_\beta > 0$. Since a , b and f are bounded, it follows that $\nabla \Psi_\beta^*$ is bounded by [1, Lemma 3.3]. In the discussion preceding Theorem 2.7 we have shown that the function $-\Psi_\beta^*$ is inf-compact. Recall that $\mathcal{A}_v = \mathcal{L} + 2\langle av, \nabla \rangle$. With $\Psi^* = \Psi_\beta^*$ we have

$$\mathcal{A}_v \Psi^* + \|v - \nabla \Psi^*\|_a^2 + F_v = -\Lambda_\beta, \quad (2.80)$$

Let χ be a convex $C^2(\mathbb{R})$ function such that $\chi(x) = x$ for $x \geq 0$, $\chi(x) = -1$ for $x \leq -1$, and χ' , χ'' are positive on $(-1, 0)$. Define $\chi_R(x) := -R + \chi(x + R)$, $R > 0$. Then we have from (2.80) that

$$-\mathcal{A}_v \chi_R(\Psi^*) + \chi_R''(\Psi^*) \|\nabla \Psi^*\|_a^2 - \chi_R'(\Psi^*) \|v - \nabla \Psi^*\|_a^2 + \chi_R'(\Psi^*) F_v = -\chi_R'(\Psi^*) \Lambda_\beta. \quad (2.81)$$

Since a and b are bounded, if $v \in \mathcal{M}$ is such that $\int_{\mathbb{R}^d} F_v d\eta_v < \infty$, then $\int_{\mathbb{R}^d} \mathcal{A}_v g(x) \eta_v(dx) = 0$ for all $g \in C_b^\infty(\mathbb{R}^d)$. This can be easily established by using a cutoff function. Thus an application of Theorem 1.1 in [13] using the property that $\int_{\mathbb{R}^d} |b + 2\langle av, \nabla \Psi^* \rangle|^2 d\eta_v < \infty$, and the fact that a is bounded and uniformly strictly elliptic, shows that η_v has a density $\rho_v \in L^{d/(d-1)}(\mathbb{R}^d)$. Therefore, as explained in the discussion preceding the theorem, since $\chi_R(\Psi^*) + R + 1$ has compact support, we have $\int_{\mathbb{R}^d} \mathcal{A}_v \chi_R(\Psi^*) \eta_v(dx) = 0$. Thus letting $R \rightarrow \infty$ in (2.81), using monotone convergence, we obtain (2.79).

We next show uniqueness. Let $\pi = \eta_v \otimes v \in \mathcal{M}$ be optimal, and $\pi_* = \eta_* \otimes v_*$ denote the ergodic occupation measure corresponding to $v_* = \nabla \Psi_\beta^*$. Here, $\eta_* = \mu_\beta^*$. Let ρ_* denote the density of η_* . Define $\bar{\eta} = \frac{1}{2}(\eta_v + \eta_*)$ and $\bar{v} = \zeta_v v + \zeta_* v_*$, with ζ_v and ζ_* given by $\zeta_v = \frac{\rho_v}{\rho_v + \rho_*}$ and $\zeta_* = \frac{\rho_*}{\rho_v + \rho_*}$, respectively. It is straightforward to verify, using the fact that the drift is affine in the control, that $\bar{\pi} = \bar{\eta} \otimes \bar{v}$ is in \mathcal{M} .

By optimality, we have

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^d} F_v(x) \eta_v(dx) + \int_{\mathbb{R}^d} F_{v_*}(x) \eta_*(dx) - 2 \int_{\mathbb{R}^d} F_{\bar{v}}(x) \bar{\eta}(dx) \\ &= 2 \int_{\mathbb{R}^d} \left(\zeta_v(x) \|v(x)\|_{a(x)}^2 + \zeta_*(x) \|v_*(x)\|_{a(x)}^2 - \|\zeta_v(x)v(x) + \zeta_*(x)v_*(x)\|_{a(x)}^2 \right) \bar{\eta}(dx) \\ &= \int_{\mathbb{R}^d} \frac{\rho_v(x) \rho_*(x)}{\rho_v(x) + \rho_*(x)} \|v(x) - v_*(x)\|_{a(x)}^2 dx. \end{aligned} \quad (2.82)$$

Since ρ_* is strictly positive, (2.82) implies that $\rho_v |v - v_*| = 0$ a.e. in \mathbb{R}^d , and thus $v = v_*$ on the support of η_v . It is clear that if v is modified outside the support of η_v then the modified $\eta_v \otimes v$ is also an infinitesimal ergodic occupation measure. Therefore $\eta_v \otimes v_* \in \mathcal{M}$. The uniqueness of the invariant measure of the diffusion with generator \mathcal{A}_{v_*} then implies that $\eta_v = \eta_*$, which in turn implies that $v = \nabla \psi_\beta^*$ a.e. in \mathbb{R}^d .

We now turn to the case $\Lambda_\beta \leq 0$ and (2.1) is recurrent. Write (2.74) as

$$\mathcal{A}_v \check{\psi} - \|v + \nabla \check{\psi}\|_a^2 + F_v = -\Lambda_\beta, \quad (2.83)$$

with $\check{\psi} = -\psi_\beta^*$. Then we have from (2.83) that

$$\mathcal{A}_v \chi_R(\check{\psi}) - \chi_R''(\check{\psi}) \|\nabla \check{\psi}\|_a^2 - \chi_R'(\check{\psi}) \|v + \nabla \check{\psi}\|_a^2 + \chi_R'(\check{\psi}) F_v = -\chi_R'(\check{\psi}) \Lambda_\beta. \quad (2.84)$$

As discussed in the paragraph preceding the theorem, the function ψ_β^* is inf-compact. We proceed exactly as before. This completes the proof. \square

Remark 2.3 The proof of Theorem 2.7 provides a general recipe to prove the lack of an optimality gap in ergodic control problems. Note that the model in [27] is such that $\nabla \psi_\beta^*$ is bounded, and a is also bounded. Therefore

$$\int \chi_R''(\check{\psi}) \|\nabla \check{\psi}\|_a^2 d\eta_v \rightarrow 0$$

as $R \rightarrow \infty$, and the proof in Theorem 2.7 goes through even for the more general Hamiltonian $H(x, p)$ considered in [27].

We should also mention here that if $\lambda_c < \Lambda_\beta \leq 0$ and (2.1) is recurrent, equation (2.79) holds even for unbounded a and b , provided b has at most affine growth, and Ψ_β^* is in $C^2(\mathbb{R}^d)$ (this is the case, for example, if b and f are locally Hölder continuous). The function ψ_β^* remains inf-compact, and thus $\int_{\mathbb{R}^d} \mathcal{A}_v \chi_R(\check{\psi}) \eta_v(dx) = 0$ since $C_c^\infty(\mathbb{R}^d)$ is locally dense in $C_c^2(\mathbb{R}^d)$. Using the inequality $\|v + \nabla \check{\psi}\|_a^2 \leq 2\|v\|_a^2 + 2\|\nabla \check{\psi}\|_a^2$, then integrating (2.84) with respect to η_v , and rearranging terms we obtain

$$\int (\chi_R''(\check{\psi}) + \chi_R'(\check{\psi})) \|\nabla \check{\psi}\|_a^2 d\eta_v \leq \int \chi_R'(\check{\psi}) \beta f d\eta_v - \Lambda_\beta \int \chi_R'(\check{\psi}) d\eta_v. \quad (2.85)$$

Thus letting $R \rightarrow \infty$ in (2.85), using monotone convergence, we obtain the energy inequality

$$\int \|\nabla \check{\psi}\|_a^2 d\eta_v \leq \int \beta f d\eta_v - \Lambda_\beta < \infty. \quad (2.86)$$

Then (2.79) follows by letting $R \rightarrow \infty$ in (2.85), using again monotone convergence and (2.86).

Remark 2.4 If $\Lambda_\beta > \lambda_c$, and under some nonanticipative control U the diffusion (2.76) has a unique weak solution, it was shown in the discussion following Remark 2.2 that $\mathcal{J}_x(U) \geq -\Lambda_\beta$, provided $\Lambda_\beta \leq 0$ and (2.1) is recurrent. The same conclusion can be drawn if $\Lambda_\beta > 0$ and under the hypotheses of Theorem 2.7. Define the set of *mean empirical measures* $\{\xi_{x,t}^U, t \geq 0\}$ of (2.76) under the control U by

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, u) \xi_{x,t}^U(dx, du) = \frac{1}{t} \int_0^t \widehat{\mathbb{E}}_x^U [h(Z_t, U_t)] dt \quad \forall h \in C_b(\mathbb{R}^d \times \mathbb{R}^d).$$

If $\Lambda_\beta > 0$, then $F_u(x) - \Lambda_\beta$ is bounded away from zero for all x outside some compact set, and one can follow the arguments in the proof of [2, Lemma 3.4.6] to show that every limit point in $\mathcal{P}(\overline{\mathbb{R}^d \times \mathbb{R}^d})$ (the set of Borel probability measures on the one-point compactification of $\mathbb{R}^d \times \mathbb{R}^d$) of a sequence of mean empirical measures $\{\xi_{x,t_n}^{U_n}, n \in \mathbb{N}\}$ as $t_n \rightarrow \infty$ takes the form $\delta\pi + (1 - \delta)\pi_\infty$, where π is an infinitesimal ergodic occupation measure and $\pi_\infty(\{\infty\}) = 1$. Using this property, one can show, by following the argument in the proof of [2, Theorem 3.4.7], that if $\mathcal{J}_x(U) \leq -\Lambda_\beta$, then the mean empirical measures are necessarily tight in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\delta = 1$ in this decomposition. This of course implies that $\mathcal{J}_x(U) = -\Lambda_\beta$. This argument establishes optimality over the largest possible class of controls U .

2.4.2 Differentiability of Λ_β

Differentiability of the map $\beta \mapsto \Lambda_\beta$ for all $\beta > \beta_c$ is established in [28, Proposition 5.4] under the hypothesis that the coefficients a, b as well as f are Lipschitz continuous and bounded in \mathbb{R}^d , but for a more general class of Hamiltonians (see (A1)–(A3) in [28]). These assumptions are used to show that $\nabla \Psi^*$ is bounded in \mathbb{R}^d , and this is utilized in the proofs.

In the next theorem we demonstrate this differentiability result for the model in this paper that assumes only measurable b and f , in which case it is not possible, in general, to obtain gradient estimates and follow the approach in [4, 27, 28]. The first assertion in this theorem should be compared to [28, Proposition 5.4].

Recall that $\tilde{\Psi}_\varepsilon := \frac{\Psi_{\beta-\varepsilon}^*}{\Psi_\beta^*}$, and $\tilde{\psi}_\varepsilon = \log \tilde{\Psi}_\varepsilon$.

Theorem 2.8 *Suppose $f \in \mathcal{B}_0^+(\mathbb{R}^d)$, and that $\beta > \beta_c$. Then for all $\varepsilon > 0$ such that $\beta - \varepsilon > \beta_c$, we have*

$$\varepsilon \frac{\mu_\beta^*(f \tilde{\Psi}_\varepsilon)}{\mu_\beta^*(\tilde{\Psi}_\varepsilon)} \leq \Lambda_\beta - \Lambda_{\beta-\varepsilon} = \mu_\beta^*(\varepsilon f - \|\nabla \tilde{\psi}_\varepsilon\|_a^2). \quad (2.87)$$

In addition, we have

$$\frac{d\Lambda_\beta}{d\beta} = \mu_\beta^*(f). \quad (2.88)$$

Proof Fix some $\varepsilon_1 > 0$ such that $\beta - 2\varepsilon_1 > \beta_c$, and consider (2.66). As argued in the proof of Theorem 2.5, the function $\tilde{\Psi}_\varepsilon$ is bounded away from 0 on \mathbb{R}^d for all $\varepsilon \in (0, \varepsilon_1]$. We recall the notation $\tilde{\mathbb{E}}^{\Psi_\beta^*}[\cdot] = \tilde{\mathbb{E}}^*[\cdot]$. Applying the Itô–Krylov formula and Fatou’s lemma to (2.66) we obtain

$$\frac{1}{T} \tilde{\mathbb{E}}_x^* \left[\int_0^T (\Lambda_{\beta-\varepsilon} - \Lambda_\beta + \varepsilon f(Y_t^*)) \tilde{\Psi}_\varepsilon(Y_t^*) dt \right] \geq 0,$$

from which the left hand side inequality of (2.87) follows by an application of Birkhoff’s ergodic theorem. Also the analogous estimate in (2.54) holds for $\tilde{\Psi}_\varepsilon$, which implies that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \tilde{\mathbb{E}}_x^* [\tilde{\Psi}_\varepsilon(Y_t^*)] = 0 \quad \forall \varepsilon \in (0, \varepsilon_1]. \quad (2.89)$$

The second equality in (2.87) follows by first using the technique in the proof of Theorem 2.5 and (2.89) to establish (2.56) for $\tilde{\Psi}_\varepsilon$, $\varepsilon \in (0, \varepsilon_1)$, and then applying the Itô–Krylov formula to the log-transformed equation corresponding to (2.66) as in (2.60) and taking limits at $t \rightarrow \infty$.

Using the convexity of $\beta \mapsto \Lambda_\beta$, we write (2.87) as

$$\frac{\mu_\beta^*(f \tilde{\Psi}_\varepsilon)}{\mu_\beta^*(\tilde{\Psi}_\varepsilon)} \leq \frac{\Lambda_\beta - \Lambda_{\beta-\varepsilon}}{\varepsilon} \leq \frac{\Lambda_{\beta+\varepsilon} - \Lambda_\beta}{\varepsilon} \leq \mu_{\beta+\varepsilon}^*(f). \quad (2.90)$$

Fix an open ball $\mathcal{B} \subset \mathbb{R}^d$, such that

$$\Lambda_{\beta-2\varepsilon_1} - \Lambda_{\beta-\varepsilon} + (2\varepsilon_1 - \varepsilon)f \leq -\delta < 0 \quad \forall \varepsilon \in [-\varepsilon_1, \varepsilon_1]. \quad (2.91)$$

This is clearly possible since $\varepsilon \mapsto \Lambda_{\beta-\varepsilon}$ is nonincreasing, $\Lambda_{\beta-2\varepsilon_1} < \Lambda_{\beta-\varepsilon_1}$, and f vanishes at infinity. Let $\tau = \tau(\mathcal{B}^c)$. Since the ground state process corresponding to $\Lambda_{\beta-\varepsilon}$ is exponentially ergodic for $\varepsilon < \varepsilon_1$ by Theorem 2.5, we have

$$\Psi_{\beta-\varepsilon}^*(x) = \mathbb{E}_x \left[e^{\int_0^\tau [(\beta-\varepsilon)f(X_s) - \Lambda_{\beta-\varepsilon}] ds} \Psi_{\beta-\varepsilon}^*(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] \quad \forall \varepsilon \in [-\varepsilon_1, \varepsilon_1] \quad (2.92)$$

by Lemma 2.7. Since $\Psi_{\beta-\varepsilon}^*$ and its inverse are bounded on \mathcal{B} , uniformly in $\varepsilon \in [-\varepsilon_1, 2\varepsilon_1]$, it follows from (2.91) and (2.92) that there exists κ such that $\Psi_{\beta-\varepsilon}^* \leq \kappa \Psi_{\beta-2\varepsilon_1}^*$ for all $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. Therefore, since the collection $\{\Psi_{\beta-\varepsilon}^*, \varepsilon \in [-\varepsilon_1, \varepsilon_1]\}$, is bounded in $C_{\text{loc}}^{1,\alpha}(\mathcal{B})$, $\alpha > 0$, we can use (2.92) and the dominated convergence theorem to conclude that

$\tilde{\Psi}_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Thus, one more application of the dominated convergence theorem shows that $\mu_\beta^*(f \tilde{\Psi}_\varepsilon) \rightarrow \mu_\beta^*(f)$ and $\mu_\beta^*(\tilde{\Psi}_\varepsilon) \rightarrow 1$ as $\varepsilon \searrow 0$. This shows that the leftmost term in (2.90) converges to $\mu_\beta^*(f)$ as $\varepsilon \searrow 0$.

We next study the term $\mu_{\beta+\varepsilon}^*(f)$. Let $\tilde{\mathbb{E}}_x^{*,\varepsilon}$ denote the expectation operator for the ground state diffusion corresponding to $\Lambda_{\beta+\varepsilon}$. Since

$$\tilde{\mathcal{L}} \Psi_{\beta+\varepsilon}^* \frac{\Psi_{\beta-2\varepsilon_1}^*}{\Psi_{\beta+\varepsilon}^*} = (\Lambda_{\beta-2\varepsilon_1} - \Lambda_{\beta+\varepsilon} - (2\varepsilon_1 + \varepsilon)f) \frac{\Psi_{\beta-2\varepsilon_1}^*}{\Psi_{\beta+\varepsilon}^*},$$

it follows by an estimate similar to (2.91) that $\tilde{\mathbb{E}}_x^{*,\varepsilon}[e^{\kappa \tau}] \leq \frac{\Psi_{\beta-2\varepsilon_1}^*}{\Psi_{\beta+\varepsilon}^*}(x)$ for all $x \in \mathcal{B}^c$ (see also Theorem 3.1 in section 3).

It is straightforward to show using this that $\inf_{\varepsilon \in [0, \varepsilon_1]} \mu_{\beta+\varepsilon}^*(\mathcal{B}) > 0$. Indeed, let $\tilde{\mathcal{B}}$ be a larger ball such that $\bar{\mathcal{B}} \subset \tilde{\mathcal{B}}$. It suffices to exhibit the result for $\tilde{\mathcal{B}}$. We have for some positive constants δ_i , $i = 1, 2, 3$, that

$$\sup_{\varepsilon \in [0, \varepsilon_1]} \sup_{x \in \partial \tilde{\mathcal{B}}} \tilde{\mathbb{E}}_x^{*,\varepsilon}[\tau] \leq \kappa^{-1} \sup_{\varepsilon \in [0, \varepsilon_1]} \sup_{x \in \partial \tilde{\mathcal{B}}} \frac{\Psi_{\beta-2\varepsilon_1}^*}{\Psi_{\beta+\varepsilon}^*}(x) =: \delta_1 < \infty,$$

and also (see [2, Theorem 2.6.1])

$$0 < \delta_2 \leq \inf_{\varepsilon \in [0, \varepsilon_1]} \sup_{x \in \partial \mathcal{B}} \tilde{\mathbb{E}}_x^{*,\varepsilon}[\tau(\tilde{\mathcal{B}}^c)] \leq \sup_{\varepsilon \in [0, \varepsilon_1]} \sup_{x \in \partial \mathcal{B}} \tilde{\mathbb{E}}_x^{*,\varepsilon}[\tau(\tilde{\mathcal{B}}^c)] \leq \delta_3 < \infty.$$

We use the inequality

$$\mu_{\beta+\varepsilon}^*(\tilde{\mathcal{B}}) \geq \frac{\delta_2}{\delta_1 + \delta_3}$$

which follows from the well known characterization of invariant probability measures due to Hasminskii [2, Theorem 2.6.9], which establishes the result.

Thus, the corresponding densities $\eta_{\beta+\varepsilon}^*$ are locally bounded and also bounded away from 0, uniformly in $\varepsilon \in [0, \varepsilon_1]$ by the Harnack inequality (see proof of equation (3.2.6) in [2]), and therefore standard pde estimates of the Fokker–Planck equation show that they are locally Hölder equicontinuous [23, Theorem 8.24, p. 202]. Given any $\theta \in (0, 1)$ we may enlarge \mathcal{B} so that $\mu_\beta^*(\mathcal{B}) \geq 1 - \theta$ and $|f| \leq \theta$ on \mathcal{B}^c . Let $\bar{\eta}_\beta$ be the (uniform) limit of $\eta_{\beta+\varepsilon_n}^*$ on \mathcal{B} along some subsequence $\varepsilon_n \searrow 0$. Since $\nabla \Psi_{\beta-\varepsilon}^*$ is Hölder equicontinuous on \mathcal{B} , uniformly in $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ as argued in the preceding paragraph, it follows that $\bar{\eta}_\beta$ is strictly positive on $\bar{\mathcal{B}}$. It is straightforward to show then that $\bar{\eta}_\beta$ is a positive solution of the Fokker–Planck equation for the (adjoint of the) operator $\mathcal{L} + 2\langle a \nabla \Psi_\beta^*, \nabla \rangle$. By the uniqueness of the invariant probability measure we have $\bar{\eta}_\beta = C \eta_\beta^*$ for some positive constant C . Since $\int_{\mathcal{B}} \bar{\eta}_\beta(x) dx \leq 1$, we have $C \leq (1 - \theta)^{-1}$. Thus, since $\sup_{\varepsilon \in [0, \varepsilon_1]} \|\eta_{\beta+\varepsilon}^*\|_\infty < \infty$, and $|f| < \theta$ on \mathcal{B}^c , by Fatou's lemma we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mu_{\beta+\varepsilon_n}^*(f) &\leq \limsup_{n \rightarrow \infty} \int_{\mathcal{B}} f(x) \eta_{\beta+\varepsilon_n}^*(x) dx + \theta \\ &\leq \int_{\mathcal{B}} f(x) \bar{\eta}_\beta(x) dx + \theta \\ &\leq (1 - \theta)^{-1} \int_{\mathcal{B}} f(x) \eta_\beta^* dx + \theta \\ &\leq (1 - \theta)^{-1} \mu_\beta^*(f) + \theta. \end{aligned}$$

Since θ can be selected arbitrarily close to 0, we obtain from (2.90) that $\lim_{\varepsilon \searrow 0} \frac{\Lambda_{\beta+\varepsilon} - \Lambda_\beta}{\varepsilon} \leq \mu_\beta^*(f)$. This establishes (2.88) using the discussion after (2.92) and completes the proof. \square

3 Exponential ergodicity and strict monotonicity of principal eigenvalues

In this section we show that exponential ergodicity of (2.1) is a sufficient condition for the strict monotonicity of the principal eigenvalue. In [26, 30] exponential ergodicity is used to obtain results similar to Theorem 2.1. The studies in [26, 30] assume C^2 regularly of a , b , and f , whereas our analysis works under much weaker hypothesis on the coefficients. Let us also remark that the regularity assumptions in [26, 30] can not be waived as their analysis heavily uses a gradient estimate (see [26, Theorem 3.1] and [30, Lemma 2.4]) which is not available for less regular coefficients. Under additional hypotheses we show in this section that $\lambda^*(f) = \mathcal{E}(f)$. Recall the definition of $\lambda''(f)$ in (2.46). It is straightforward to show that $\lambda''(f) \geq \mathcal{E}(f)$. We present an example where $\lambda^*(f) < \mathcal{E}(f)$, and therefore also $\lambda^*(f) < \lambda''(f)$.

Example 3.1 Let $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a smooth function which is strictly positive on $[-1, 1]$ and satisfies $\phi(x) = e^{-\frac{1}{2}|x|}$ for $|x| \geq 1$. Define

$$f(x) := -\frac{1}{\phi(x)} (\phi''(x) + \text{sign}(x) \phi'(x) - \phi(x)).$$

Then $f(x) = \frac{5}{4}$ for $|x| \geq 1$, and

$$\phi''(x) + \text{sign}(x) \phi'(x) + f(x)\phi(x) = \phi(x). \quad (3.1)$$

Consider the one-dimensional controlled diffusion

$$dX_t = \text{sign}(X_t) dt + \sqrt{2} dW_t. \quad (3.2)$$

From (3.1) and Lemma 2.2 (ii) we have $\lambda^*(f) \leq 1$. It is clear that (3.2) is a transient process. Therefore for any initial data x

$$\mathcal{E}_x(f) \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T f(X_t) dt \right] = \frac{5}{4}.$$

Therefore $\lambda^*(f) < \mathcal{E}(f)$.

Remark 3.1 Example 3.1 presents a case where the conclusion of [8, Theorem 1.9] fails to hold. Since the operator \mathcal{L} in this example is uniformly elliptic, has bounded coefficients, and $d = 1$, the only aspect that makes it different from the class of operators in part (i) of [8, Theorem 1.9], is that it is not self-adjoint.

Let us start with the following result on exponential ergodicity.

Theorem 3.1 *The following are equivalent.*

- (a) *For some ball \mathcal{B}_o there exists $\delta_o > 0$ and $x_o \in \bar{\mathcal{B}}_o^c$ such that $\mathbb{E}_{x_o}[e^{\delta_o \tau(\mathcal{B}_o^c)}] < \infty$.*
- (b) *For any ball \mathcal{B} there exists $\delta > 0$ such that $\mathbb{E}_x[e^{\delta \tau(\mathcal{B}^c)}] < \infty$ for all $x \in \mathcal{B}^c$.*
- (c) *For every ball \mathcal{B} , there exists a positive function $\mathcal{V} \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, and positive constants κ_0 and δ such that*

$$\mathcal{L}\mathcal{V}(x) \leq \kappa_0 \mathbf{1}_{\mathcal{B}}(x) - \delta \mathcal{V}(x) \quad \forall x \in \mathbb{R}^d. \quad (3.3)$$

- (d) *For every ball \mathcal{B} , $\lambda^*(\mathbf{1}_{\mathcal{B}^c}) < 1$.*

Proof We first show that (a) \Rightarrow (d). It is enough to prove (d) for $\mathcal{B} \subset \mathcal{B}_o$. Let $f = \mathbf{1}_{\mathcal{B}^c}$, and consider the Dirichlet eigensolutions $(\hat{\Psi}_n, \hat{\lambda}_n)$ in (2.4). It is easy to see that $\hat{\lambda}_n < 1$ for all n . We claim that $\lambda^*(f) < 1$. If not, then $\hat{\lambda}_n \nearrow 1$ as $n \rightarrow \infty$, and $\hat{\Psi}_n$ converges to some $\Psi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p \geq d$, which satisfies $\mathcal{L}\Psi = \mathbf{1}_{\mathcal{B}}\Psi$ on \mathbb{R}^d and $\Psi(0) = 1$. The same argument used in the proof of Lemma 2.2 then shows that $\Psi(x) = \mathbb{E}_x[\Psi(X_{\tau(\mathcal{B}_o^c)})]$. Therefore Ψ attains a maximum on $\bar{\mathcal{B}}_o$, and by the strong maximum principle it must be constant. Thus $\mathcal{L}\Psi = 0$ which contradicts the fact that $\Psi(0) = 1$.

Next we show that (d) \Rightarrow (c). If $\lambda^*(f) < 1$, for $f = \mathbf{1}_{\mathcal{B}^c}$, then any limit point Ψ of the Dirichlet eigenfunctions $\widehat{\Psi}_n$ as $n \rightarrow \infty$ satisfies

$$\begin{aligned} \mathcal{L}\Psi &= \mathbf{1}_{\mathcal{B}}\Psi - (1 - \lambda^*(f))\Psi \\ &\leq \left(\sup_{\mathcal{B}} \Psi\right)\mathbf{1}_{\mathcal{B}} - (1 - \lambda^*(f))\Psi, \end{aligned}$$

and thus (c) holds with $\delta = 1 - \lambda^*(f)$. Also by [1, Lemma 2.1 (c)], we have $\inf_{\mathbb{R}^d} \Psi = \min_{\mathcal{B}} \Psi > 0$.

That (c) \Rightarrow (b) is well known, and can be shown by a standard application of the Itô–Krylov formula to (3.3), by which we obtain

$$\begin{aligned} \left(\inf_{\mathbb{R}^d} \mathcal{V}\right) \mathbb{E}_x \left[e^{\delta(\tau(\mathcal{B}^c) \wedge \tau_R)} \right] - \mathcal{V}(x) &\leq \mathbb{E}_x \left[e^{\delta(\tau(\mathcal{B}^c) \wedge \tau_R)} \mathcal{V}(X_{\tau(\mathcal{B}^c) \wedge \tau_R}) \right] - \mathcal{V}(x) \\ &\leq \mathbb{E}_x \left[\int_0^{\tau(\mathcal{B}^c) \wedge \tau_R} \left(\delta e^{\delta t} \mathcal{V}(X_t) + e^{\delta t} \mathcal{L}\mathcal{V}(X_t) \right) dt \right] \leq 0. \end{aligned}$$

The result then follows by letting $R \rightarrow \infty$, and this completes the proof. \square

We introduce the following hypothesis.

(H2) There exists a lower-semicontinuous, inf-compact function $\ell : \mathbb{R}^d \rightarrow [0, \infty)$ such that $\mathcal{E}(\ell) < \infty$, where $\mathcal{E}(\cdot)$ is as defined in (1.3).

Lemma 3.1 *Under (H2), we have $\mathcal{E}(\ell) = \lambda^*(\ell)$, and there exists a positive $V \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p \geq d$, with $\inf_{\mathbb{R}^d} V > 0$, and $V(0) = 1$, satisfying*

$$\mathcal{L}V + \ell V = \lambda^*(\ell)V \quad \text{a.e. in } \mathbb{R}^d. \quad (3.4)$$

In particular, the unique strong solution of (2.1) is exponentially ergodic.

Proof By (2.3) we have

$$\mathcal{E}_x(\ell) \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T \ell(X_s) ds \right]. \quad (3.5)$$

Since $\mathcal{E}(\ell) = \inf_{\mathbb{R}^d} \mathcal{E}_x(\ell)$, (3.5) implies that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T \ell(X_s) ds \right] < \infty$$

for some $x \in \mathbb{R}^d$. The inf-compactness of ℓ then implies that the unique strong solution of (2.1) is recurrent. That $\mathcal{E}(\ell) = \lambda^*(\ell)$, and the existence of a solution V then follow by Theorem 1.4 and Lemma 2.1 in [1], respectively. Exponential ergodicity then follows from (3.4), using Theorem 3.1. \square

An application of Itô–Krylov’s formula to (3.4), followed by Fatou’s lemma, gives

$$\mathbb{E}_x \left[e^{\int_0^{\tau_r} [\ell(X_t) - \lambda^*(\ell)] dt} V(X_{\tau_r}) \right] \leq V(x) \quad \forall x \in B_r^c, \forall r > 0, \quad (3.6)$$

where τ_r , as defined earlier, denotes the first hitting time of the ball B_r .

The next result shows that (H2) implies (P1).

Theorem 3.2 *Assume (H2). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally bounded measurable function such that $\text{ess inf}_{\mathbb{R}^d} f > -\infty$, and $\ell - f$ is inf-compact.*

Then for any continuous $h \in C_0^+(\mathbb{R}^d)$ we have

$$\lambda^*(f - h) < \lambda^*(f) = \mathcal{E}_x(f) \quad \forall x \in \mathbb{R}^d.$$

Proof Let $h \in C_0^+(\mathbb{R}^d)$, and $\tilde{f} := f - h$. It is easy to see that $\mathcal{E}(f)$ and $\mathcal{E}(\tilde{f})$ are both finite. It is shown in [1, 11] that the Dirichlet eigensolutions $(\hat{\Psi}_r, \hat{\lambda}_r)$ in (2.4) converge, along some subsequence as $r \rightarrow \infty$, to $(\Psi^*, \lambda^*(f))$ which satisfies

$$\mathcal{L}\Psi^* + f\Psi^* = \lambda^*(f)\Psi^* \quad \text{on } \mathbb{R}^d, \quad \text{and} \quad \lambda^*(f) \leq \mathcal{E}_x(f) \quad \forall x \in \mathbb{R}^d. \quad (3.7)$$

It is also clear that Lemma 2.2 (i) holds for $(\hat{\Psi}_n, \hat{\lambda}_n)$. Now choose a bounded ball \mathcal{B} such that

$$(|f(x)| + |h(x)|) + \sup_n (|\hat{\lambda}_n(f)| + |\hat{\lambda}_n(\tilde{f})|) + \lambda^*(\ell) + 1 < \ell(x) \quad \forall x \in \mathcal{B}.$$

This is possible since $\ell - f$ is inf-compact. In view of (3.6) we note that (2.11) holds with $(f - h - \lambda^*(f - h))$ replaced by $\ell - \lambda^*(\ell)$. Thus with the above choice of \mathcal{B} , we can justify the passing to the limit in (2.13), and therefore we obtain

$$\Psi^*(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} [f(X_t) - \lambda^*(f)] dt} \Psi^*(X_{\tau}) \right] \quad \forall x \in \mathcal{B}^c, \quad (3.8)$$

with $\tau = \tau(\mathcal{B}^c)$. A similar argument also gives

$$\tilde{\Psi}^*(x) = \mathbb{E}_x \left[e^{\int_0^{\tau} [\tilde{f}(X_t) - \lambda^*(\tilde{f})] dt} \tilde{\Psi}^*(X_{\tau}) \right] \quad \forall x \in \mathcal{B}^c. \quad (3.9)$$

In fact, the above relations hold for any bounded domain $D \supset \mathcal{B}$ with $\tau = \tau(D^c)$. Suppose that $\lambda^*(f) = \lambda^*(\tilde{f})$. Then

$$f(x) - h(x) - \lambda^*(\tilde{f}) \leq f(x) - \lambda^*(f).$$

Thus if we multiply Ψ^* with a suitable positive constant such that $\Psi^* - \tilde{\Psi}^*$ is nonnegative in \mathcal{B} and attains a minimum of 0 in \mathcal{B} , it follows from (3.8) and (3.9) that $\Psi^* - \tilde{\Psi}^*$ is nonnegative in \mathbb{R}^d . Since (2.34) holds, we conclude exactly as in the proof of Theorem 2.2 that $\lambda^*(\tilde{f}) < \lambda^*(f)$.

Next we show that $\lambda^*(f) = \mathcal{E}_x(f)$ for all $x \in \mathbb{R}^d$. We have already established the strict monotonicity of $\lambda^*(f)$ at f , and therefore Theorem 2.1 applies. Hence for any continuous g with compact support we have from [33, Theorem 1.3.10] that

$$\mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda^*(f)] dt} g(X_T) \right] = \Psi^*(x) \mathbb{E}_x^{\Psi^*} \left[\frac{g(Y_T^*)}{\Psi^*(Y_T^*)} \right] \xrightarrow{T \rightarrow \infty} \Psi^*(x) \mu^* \left(\frac{g}{\Psi^*} \right) > 0, \quad (3.10)$$

where μ^* denotes the invariant measure of the twisted process Y^* satisfying (2.25). Let $\tilde{\mathcal{B}}$ be a ball such that $f(x) - \lambda^*(f) < \ell(x) - \lambda^*(\ell)$ for $x \in \tilde{\mathcal{B}}^c$. Thus from (3.4) we obtain

$$\mathcal{L}V + (f - \lambda^*(f))V \leq \kappa \mathbf{1}_{\tilde{\mathcal{B}}}, \quad (3.11)$$

with $\kappa = \max_{\tilde{\mathcal{B}}} (|f| + |\ell| + |\lambda^*| + \lambda^*(\ell)) V$. Applying Itô–Krylov’s formula to (3.11) followed by Fatou’s lemma we obtain

$$\begin{aligned} \left(\min_{\mathbb{R}^d} V \right) \mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda^*(f)] dt} \right] &\leq \mathbb{E}_x \left[e^{\int_0^T [f(X_t) - \lambda^*(f)] dt} V(X_T) \right] \\ &\leq \kappa \int_0^T \mathbb{E}_x \left[e^{\int_0^t [f(X_s) - \lambda^*(f)] ds} \mathbf{1}_{\tilde{\mathcal{B}}}(X_t) \right] dt + V(x) \\ &\leq \kappa' T + V(x), \end{aligned}$$

for some constant κ' , where in the last inequality we have used (3.10). Taking logarithms on both sides of the preceding inequality, then dividing by T , and letting $T \rightarrow \infty$, we obtain $\lambda^*(f) \geq \mathcal{E}_x(f)$ for all $x \in \mathbb{R}^d$. Combining this with (3.7) results in equality. \square

Remark 3.2 Continuity of h is superfluous in Theorem 3.2. The result holds if h is a non-trivial, nonnegative measurable function, vanishing at infinity.

Corollary 3.1 Assume (H2). Then for every $\tilde{f} \not\leq f$, such that $\text{ess inf}_{\mathbb{R}^d} \tilde{f} > -\infty$ and $(\ell - f)$ is inf-compact, we have $\lambda^*(\tilde{f}) < \lambda^*(f)$.

Proof Note that for any cut-off function χ we have $\lambda^*(\tilde{f}) \leq \lambda^*(\chi\tilde{f} + (1-\chi)f)$. Then the result follows from [Theorem 3.2](#) and [Remark 3.2](#). \square

Remark 3.3 In [Theorem 3.1](#) we can replace the assumption on f by the hypothesis that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a locally bounded measurable function and $\ell - |f|$ is inf-compact. The same applies to [Theorem 3.4](#) that appears later in this section. Note that in this case f need not be bounded below.

Let us now discuss the exponential ergodicity and show that this implies (H2).

Proposition 3.1 Let $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$ be inf-compact, and suppose $\phi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ is bounded below in \mathbb{R}^d and satisfies

$$\mathcal{L}\phi + \langle \nabla\phi, a\nabla\phi \rangle = -\ell. \quad (3.12)$$

then $\mathcal{E}_x(\ell) < \infty$ for all $x \in \mathbb{R}^d$.

Proof Let $\Phi(x) = \exp(\phi(x))$. Then $\inf_{\mathbb{R}^d} \Phi > 0$ and (3.12) gives

$$\mathcal{L}\Phi + \ell\Phi = 0. \quad (3.13)$$

Now apply Itô–Krylov’s formula to (3.13) followed by Fatou’s lemma to obtain

$$\mathbb{E}_x \left[e^{\int_0^T \ell(X_s) ds} \Phi(X_T) \right] \leq \Phi(x).$$

Taking logarithm on both sides, diving by T and letting $T \rightarrow \infty$, we obtain $\mathcal{E}_x(\ell) < \infty$. \square

Example 3.2 Let $a = \frac{1}{2}I$ and $b(x) = b_1(x) + B(x)$ where B is bounded and

$$\langle b_1(x), x \rangle \leq -\kappa|x|^\alpha, \quad \text{for some } \alpha \in (1, 2].$$

Then we take $\phi(x) = \theta|x|^\alpha$ for $|x| \geq 1$, $\theta \in (0, 1)$. It is easy to check that for a suitable choice of $\theta \in (0, 1)$, (3.12) holds for $\ell(x) \sim |x|^{2\alpha-2}$.

Remark 3.4 Equation (3.12) is a stronger condition than the strict monotonicity of $\lambda^*(f)$ at f . In fact, (3.12) might not hold in many important situations. For instance, if a, b are both bounded and a is uniformly elliptic, then it is not possible to find inf-compact ℓ satisfying (3.12). Otherwise we can find a finite principal eigenvalue for the operator \mathcal{L}^ℓ , by a same method as in (3.7), which would contradict [8, Proposition 2.6].

Even though (3.12) does not hold for bounded a and b , strict monotonicity of $\lambda^*(f)$ at f can be asserted under suitable hypotheses. This is the subject of the following theorem.

Theorem 3.3 Let $\mathcal{V} \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ be a positive function with $\inf_{\mathbb{R}^d} \mathcal{V} > 0$ satisfying

$$\mathcal{L}\mathcal{V} \leq \kappa_0 \mathbf{1}_{\mathcal{K}} - \gamma\mathcal{V} \quad \text{on } \mathbb{R}^d, \quad (3.14)$$

for some compact set \mathcal{K} and positive constants κ_0 and γ . Let f be a nonnegative bounded measurable function with $\limsup_{x \rightarrow \infty} f(x) < \gamma$. Then for any $h \in C_0^+(\mathbb{R}^d)$, we have $\lambda^*(f - h) < \lambda^*(f) = \mathcal{E}_x(f)$.

Proof Let $\tilde{f} = f - h$. Suppose $\lambda^*(\tilde{f}) = \lambda^*(f)$. Applying an argument similar to (3.7) we can find $\Psi^*, \tilde{\Psi}^*$ that satisfies

$$\begin{aligned}\mathcal{L}\Psi^* + f\Psi^* &= \lambda^*(f)\Psi^*, \\ \mathcal{L}\tilde{\Psi}^* + (f-h)\tilde{\Psi}^* &= \lambda^*(\tilde{f})\tilde{\Psi}^*.\end{aligned}$$

Let $\mathcal{K}_0 \supset \mathcal{K}$ be any compact set such that $f < \gamma$ on \mathcal{K}_0^c . If τ denotes the first hitting time to the compact set \mathcal{K}_0 , then by an application of Itô–Krylov’s formula to (3.14) we obtain

$$\mathbb{E}_x[e^{\gamma\tau}] < \infty, \quad x \in \mathcal{K}_0^c.$$

We next use the fact that if \mathcal{L} corresponds to a recurrent diffusion and f is nonnegative then $\lambda^*(f) \geq 0$. Indeed, in this case we have $\mathcal{L}\Psi^* \leq \lambda^*(f)\Psi^*$. If $\lambda^*(f) \leq 0$, this implies that $\Psi^*(X_t)$ is a nonnegative supermartingale and since it is integrable, it converges a.s. Since the process is recurrent, this implies that Ψ^* must equal to a constant, which, in turn, necessitates that $\lambda^*(f) = 0$ (and $f = 0$). Since $\lambda^*(f) \geq 0$, an argument similar to the proof of Lemma 2.2 (ii) gives that for $x \in \mathcal{K}_0^c$,

$$\begin{aligned}\Psi^*(x) &= \mathbb{E}_x\left[e^{\int_0^\tau [f(X_t) - \lambda^*(f)] dt} \Psi^*(X_\tau)\right], \\ \tilde{\Psi}^*(x) &= \mathbb{E}_x\left[e^{\int_0^\tau [\tilde{f}(X_t) - \tilde{\lambda}^*(\tilde{f})] dt} \tilde{\Psi}^*(X_\tau)\right].\end{aligned}$$

Therefore applying the strong maximum principle as in Theorem 3.2, we obtain $h\tilde{\Psi}^* = 0$ which is a contradiction since $h \neq 0$ and $\tilde{\Psi}^* > 0$. Thus we have $\lambda^*(f-h) < \lambda^*(f)$. The proof of $\lambda^*(f) = \mathcal{E}_x(f)$ follows from a similar argument as in Theorem 3.2. \square

Example 3.3 Suppose $a = \frac{1}{2}I$, where I denotes the identity matrix, and

$$b(x) \cdot x \leq -|x|, \quad \text{outside a compact set } \mathcal{K}_1.$$

With $\mathcal{V}(x) = \exp(|x|)$ for $|x| \geq 1$, we have

$$\mathcal{L}\mathcal{V} = \left(\frac{d-1}{2|x|} + \frac{1}{2} - b(x) \cdot \frac{x}{|x|}\right)\mathcal{V} \leq \left(\frac{d-1}{2|x|} - \frac{1}{2}\right)\mathcal{V} \quad \text{for } |x| \geq 1.$$

We next discuss the property known as *minimal growth at infinity* [8, Definition 8.2]. As shown in [8, Proposition 8.4], minimal growth at infinity implies that the eigenspace corresponding to the eigenvalue $\lambda^*(f)$ is one dimensional, i.e., $\lambda^*(f)$ is simple. Recall the definition of the generalized principal eigenvalue in the domain D . Define

$$\lambda_1(D) = \inf \left\{ \lambda : \exists \varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d), \varphi > 0, \mathcal{L}\varphi + (f - \lambda)\varphi \leq 0 \text{ a.e. in } D \right\}.$$

Note that $\lambda_1(\mathbb{R}^d) = \hat{\Lambda}(f) = \lambda^*(f)$. It is also clear from this definition that for $D_1 \subset D_2$ we have $\lambda_1(D_1) \leq \lambda_1(D_2)$.

Definition 3.1 (Minimal growth at infinity) A positive function $\varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ satisfying

$$\mathcal{L}\varphi + (f - \lambda^*(f))\varphi = 0, \quad \text{a.e. in } \mathbb{R}^d \tag{3.15}$$

is said to be a solution of (3.15) of minimal growth at infinity if for any $r > 0$ and any positive function $v \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d \setminus B_r)$ satisfying $\mathcal{L}v + (f - \lambda^*(f))v \leq 0$ a.e., in B_r^c , there exists $R > r$ and $k > 0$ such that $k\varphi \leq v$ in B_R^c .

The above definition differs slightly from [8, Definition 8.2]. We have stated it in a form that suits us. We have already seen that under (H2) we have a unique positive eigenfunction. Next we show that (H2) in fact implies the minimal growth at infinity property for the operator $\mathcal{L}^{(f-\lambda^*)}$. This extends [8, Corollary 8.6].

Theorem 3.4 Assume (H2). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be locally bounded measurable function such that $\text{essinf } f > -\infty$, and $(\ell - f)$ be inf-compact. Then the principal eigenfunction Ψ^* satisfying (3.15) is of minimal growth at infinity.

Proof In view of [8, Theorem 8.5] it is enough to show that

$$\lambda_1(\mathbb{R}^d) > \lim_{r \rightarrow \infty} \lambda_1(B_r^c).$$

Since $r \mapsto \lambda_1(B_r^c)$ is non-increasing this follows if we show that $\lambda_1(\mathbb{R}^d) > \lambda_1(B_1^c)$. We argue by contradiction. Suppose that $\lambda_1(\mathbb{R}^d) = \lambda_1(B_1^c) = \lambda_1$. By [8, Theorem 1.4] there exists $\tilde{\Psi} \in \mathcal{W}_{\text{loc}}^{2,p}(B_1^c) \cap C(\mathbb{R}^d)$, $p > d$, satisfying

$$\mathcal{L}\tilde{\Psi} + (f - \lambda_1)\tilde{\Psi} = 0, \quad \tilde{\Psi} > 0 \text{ in } \overline{B_1^c}, \quad \text{and } \tilde{\Psi} = 0 \text{ on } \partial B_1.$$

In particular, as shown in [8, Proof of Theorem 1.4, Step 1], such a solution can be obtained as a limit of

$$\mathcal{L}\tilde{\Psi}_n + (f - \lambda_1(B_n \setminus B_1))\tilde{\Psi}_n = 0, \quad \tilde{\Psi}_n > 0 \text{ in } B_n \setminus \overline{B_1^c}, \quad \text{and } \tilde{\Psi}_n = 0 \text{ on } \partial(B_n \cup B_1).$$

Let τ denote the first hitting time of some open ball $\mathcal{B} \supset \overline{B_1}$. Then, following the same argument as in the proof Lemma 2.2, we obtain

$$\tilde{\Psi}(x) = \mathbb{E}_x \left[e^{\int_0^\tau [f(X_s) - \lambda_1] ds} \tilde{\Psi}(X_\tau) \right], \quad \text{for } x \in \mathcal{B}^c.$$

We can choose the compact \mathcal{B} large enough so that

$$\Psi^*(x) = \mathbb{E}_x \left[e^{\int_0^\tau [f(X_s) - \lambda_1] ds} \Psi^*(X_\tau) \right], \quad \text{for } x \in \mathcal{B}^c,$$

where Ψ^* is the principal eigenfunction in \mathbb{R}^d . Scale Ψ^* by multiplying it with a positive constant so that $\Psi^* - \tilde{\Psi}$ is nonnegative in $\overline{\mathcal{B}} \setminus B_1$ and its minimum in this set equals 0. Thus by the stochastic representations above we have $\Psi^* - \tilde{\Psi} \geq 0$. In addition,

$$\mathcal{L}(\Psi^* - \tilde{\Psi}) - (f - \lambda_1)^-(\Psi^* - \tilde{\Psi}) = -(f - \lambda_1)^+(\Psi^* - \tilde{\Psi}) \leq 0.$$

By the strong maximum principle, we obtain $\Psi^* - \tilde{\Psi} = 0$ in $\overline{B_1^c}$, which is not possible as $\Psi^* - \tilde{\Psi} > 0$ on ∂B_1 . This contradicts the original hypothesis, and the proof is complete. \square

4 Risk-sensitive control

In this section we apply the results developed in the previous sections to the risk-sensitive control problem. As mentioned earlier, we completely characterize the optimal Markov control for the risk-sensitive control problem in this section. In addition, we also establish uniqueness of the value function (see Theorems 4.1 and 4.2). Another interesting result is the continuity of the controlled principal eigenvalue with respect to the stationary Markov controls. This is done in Theorem 4.3. We first introduce the control problem.

4.1 The controlled diffusion model

We consider a controlled diffusion process $X = \{X_t, t \geq 0\}$ which takes values in the d -dimensional Euclidean space \mathbb{R}^d , and is governed by the Itô stochastic differential equation

$$dX_t = b(X_t, U_t) dt + \sigma(X_t) dW_t. \quad (4.1)$$

All random processes in (4.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d -dimensional standard Wiener process independent of the initial condition X_0 . The control process U takes values in a compact, metrizable set \mathbb{U} , and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. The set \mathcal{U} of *admissible controls* consists of the control processes U that are *non-anticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s := \text{the completion of } \bigcap_{y>s} \sigma\{X_0, U_r, W_r, r \leq y\} \text{ relative to } (\mathfrak{F}, \mathbb{P}).$$

We impose the following standard assumptions on the drift b and the diffusion matrix σ to guarantee existence and uniqueness of solutions.

(B1) *Local Lipschitz continuity*: The functions $b: \mathbb{R}^d \times \mathbb{U} \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ are continuous, and satisfy

$$|b(x, u) - b(y, u)| + \|\sigma(x) - \sigma(y)\| \leq C_R |x - y| \quad \forall x, y \in B_R, \forall u \in \mathbb{U}.$$

for some constant $C_R > 0$ depending on $R > 0$.

(B2) *Affine growth condition*: b and σ satisfy

$$\sup_{u \in \mathbb{U}} \langle b(x, u), x \rangle^+ + \|\sigma(x)\|^2 \leq C_0(1 + |x|^2) \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_0 > 0$.

(B3) *Nondegeneracy*: Assumption (A3) in [subsection 1.1](#) holds.

It is well known that under (B1)–(B3), for any admissible control there exists a unique solution of (4.1) [2, Theorem 2.2.4]. We define the family of operators $\mathcal{L}_u: C^2(\mathbb{R}^d) \mapsto C(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$\mathcal{L}_u f(x) := a^{ij}(x) \partial_{ij} f(x) + b^i(x, u) \partial_i f(x), \quad u \in \mathbb{U}.$$

4.1.1 The risk-sensitive criterion

Let \mathfrak{C} denote the class of functions $c(x, u)$ in $C(\mathbb{R}^d \times \mathbb{U}, \mathbb{R}_+)$ that are locally Lipschitz in x uniformly with respect to $u \in \mathbb{U}$. We let $c \in \mathfrak{C}$ denote the *running cost* function, and for any admissible control $U \in \mathfrak{U}$, we define the risk-sensitive objective function $\mathcal{E}_x^U(c)$ by

$$\mathcal{E}_x^U(c) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x \left[e^{\int_0^T c(X_s, U_s) ds} \right]. \quad (4.2)$$

We also define $\Lambda_x^* := \inf_{U \in \mathfrak{U}} \mathcal{E}_x^U(c)$.

4.2 Relaxed controls

We adopt the well known *relaxed control* framework [2]. According to this relaxation, a stationary Markov control is a measurable map from \mathbb{R}^d to $\mathcal{P}(\mathbb{U})$, the latter denoting the set of probability measures on \mathbb{U} under the Prokhorov topology. Let \mathfrak{U}_{SM} denote the class of all such stationary Markov controls. A control $\nu \in \mathfrak{U}_{\text{SM}}$ may be viewed as a kernel on $\mathcal{P}(\mathbb{U}) \times \mathbb{R}^d$, which we write as $\nu(du|x)$. We say that a control $\nu \in \mathfrak{U}_{\text{SM}}$ is precise if it is a measurable map from \mathbb{R}^d to \mathbb{U} . We extend the definition of b and c as follows. For $\nu \in \mathfrak{U}_{\text{SM}}$ we let

$$b_\nu(x) := \int_{\mathbb{U}} b(x, u) \nu(du|x), \quad \text{and} \quad c_\nu(x) := \int_{\mathbb{U}} c(x, u) \nu(du|x). \quad \text{for } \nu \in \mathcal{P}(\mathbb{U}).$$

It is easy to see from (B2) and Jensen's inequality that

$$\sup_{\nu \in \mathfrak{U}_{\text{SM}}} \langle b_\nu(x), x \rangle^+ \leq C_0(1 + |x|^2), \quad x \in \mathbb{R}^d.$$

For $\nu \in \mathfrak{U}_{\text{SM}}$, consider the relaxed diffusion

$$dX_t = b_\nu(X_t) dt + \sigma(X_t) dW_t. \quad (4.3)$$

It is well known that under $\nu \in \mathfrak{U}_{\text{SM}}$ (4.3) has a unique strong solution [24]. Moreover, under $\nu \in \mathfrak{U}_{\text{SM}}$, the process X is strong Markov, and we denote its transition kernel by $P_\nu^t(x, \cdot)$. It also follows from the work in [14] that under $\nu \in \mathfrak{U}_{\text{SM}}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus \mathcal{L}_ν defined by

$$\mathcal{L}_\nu f(x) := a^{ij}(x) \partial_{ij} f(x) + b_\nu^i(x) \partial_i f(x), \quad \nu \in \mathfrak{U}_{\text{SM}},$$

for $f \in C^2(\mathbb{R}^d)$, is the generator of a strongly-continuous semigroup on $C_b(\mathbb{R}^d)$, which is strong Feller. We let \mathbb{P}_x^v denote the probability measure and \mathbb{E}_x^v the expectation operator on the canonical space of the process under the control $v \in \mathcal{U}_{\text{SM}}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$. We denote by \mathcal{U}_{SSM} the subset of \mathcal{U}_{SM} that consists of *stable controls*, i.e., under which the controlled process is positive recurrent, and by μ_v the invariant probability measure of the process under the control $v \in \mathcal{U}_{\text{SSM}}$.

Definition 4.1 For $v \in \mathcal{U}_{\text{SM}}$ and a locally bounded measurable function $f: \mathbb{R}^d \rightarrow \mathbb{R}$, we let $\lambda_v^*(f)$ denote the principal eigenvalue of the operator $\mathcal{L}_v^f := \mathcal{L}_v + f$ on \mathbb{R}^d (see Definition 2.1).

We also adapt the notation in (2.3) to the control setting, and define

$$\mathcal{E}_x^v(f) := \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^v \left[e^{\int_0^T f(X_s) ds} \right], \quad \text{and} \quad \mathcal{E}^v(f) := \inf_{x \in \mathbb{R}^d} \mathcal{E}_x^v(f), \quad v \in \mathcal{U}_{\text{SM}}.$$

We refer to $\mathcal{E}^v(f)$ as the *risk-sensitive average* of f under the control v .

Recall the risk-sensitive objective function \mathcal{E}_x^U defined in (4.2) and the optimal value Λ^* . We say that a stationary Markov control $v \in \mathcal{U}_{\text{SM}}$ is optimal (for the risk-sensitive criterion) if $\mathcal{E}_x^v(c_v) = \Lambda^*$ for all $x \in \mathbb{R}^d$, and we let $\mathcal{U}_{\text{SM}}^*$ denote the class of these controls.

4.3 Optimal Markov controls and the risk-sensitive HJB

We start with the following assumption.

Assumption 4.1 (uniform exponential ergodicity) There exists an inf-compact function $\ell \in C(\mathbb{R}^d)$ and a positive function $\mathcal{V} \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$, satisfying $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, such that

$$\sup_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V} \leq \bar{\kappa} \mathbf{1}_{\mathcal{K}} - \ell \mathcal{V} \quad \text{a.e. on } \mathbb{R}^d, \quad (4.4)$$

for some constant $\bar{\kappa}$, and a compact set \mathcal{K} .

It is easy to see that for $\bar{\kappa}_0 := \frac{\bar{\kappa}}{\min_{\mathbb{R}^d} \mathcal{V}}$ we obtain from (4.4) that

$$\sup_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V} + (\ell - \bar{\kappa}_0) \mathcal{V} \leq 0,$$

and therefore, applying Itô–Krylov’s formula we have $\mathcal{E}^v(\ell) \leq \bar{\kappa}_0$ for any stationary Markov control $v \in \mathcal{U}_{\text{SM}}$.

Example 4.1 Let σ be bounded and $b: \mathbb{R}^d \times \mathbb{U} \rightarrow \mathbb{R}^d$ be such that

$$\langle b(x, u) - b(0, u), x \rangle \leq -\kappa |x|^\alpha, \quad \text{for some } \alpha \in (1, 2], \quad (x, u) \in \mathbb{R}^d \times \mathbb{U}.$$

Then as seen in Example 3.2, $\mathcal{V}(x) = \exp(\theta |x|^\alpha)$, for $|x| \geq 1$, satisfies (4.4) for sufficiently small $\theta > 0$, and $\ell(x) \sim |x|^{2\alpha-2}$. Note that $\alpha = 2$ and $\sigma = I$ is considered in [20].

We introduce the class of running costs \mathcal{C}_ℓ defined by

$$\mathcal{C}_\ell := \left\{ c \in \mathfrak{C} : \ell(\cdot) - \max_{u \in \mathbb{U}} c(\cdot, u) \text{ is inf-compact} \right\}.$$

The first important result of this section is the following.

Theorem 4.1 Suppose [Assumption 4.1](#) holds, and $c \in \mathcal{C}_\ell$. Then $\Lambda^* = \Lambda_x^*$ does not depend on x , and there exists a positive solution $V \in C^2(\mathbb{R}^d)$ satisfying

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u V + c(\cdot, u)V] = \Lambda^* V \quad \text{on } \mathbb{R}^d, \quad \text{and } V(0) = 1. \quad (4.5)$$

In addition, if $\overline{\mathfrak{U}}_{\text{SM}} \subset \mathfrak{U}_{\text{SM}}$ denotes the class of Markov controls v which satisfy

$$\mathcal{L}_v V + c_v V = \min_{u \in \mathbb{U}} [\mathcal{L}_u V + c(\cdot, u)V] \quad \text{a.e. in } \mathbb{R}^d,$$

the following hold.

- (a) $\overline{\mathfrak{U}}_{\text{SM}} \subset \mathfrak{U}_{\text{SM}}^*$, and it holds that $\lambda_v^*(c_v) = \Lambda^*$ for all $v \in \overline{\mathfrak{U}}_{\text{SM}}$;
- (b) $\mathfrak{U}_{\text{SM}}^* \subset \overline{\mathfrak{U}}_{\text{SM}}$;
- (c) [Equation \(4.5\)](#) has a unique positive solution in $C^2(\mathbb{R}^d)$ (up to a multiplicative constant).

Proof Using a standard argument (see [\[1, 10, 11\]](#)) we can find a pair $(V, \hat{\lambda}) \in C^2(\mathbb{R}^d) \times \mathbb{R}$, with $V > 0$ on \mathbb{R}^d , and $V(0) = 1$, that satisfies

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u V + c(\cdot, u)V] = \hat{\lambda} V, \quad \hat{\lambda} \leq \inf_{x \in \mathbb{R}^d} \Lambda_x^*. \quad (4.6)$$

This is obtained as a limit of Dirichlet eigensolutions $(\hat{V}_n, \hat{\lambda}_n) \in (\mathcal{W}_{\text{loc}}^{2,p}(B_n) \cap C(\bar{B}_n)) \times \mathbb{R}$, for any $p > d$, satisfying $\hat{V}_n > 0$ on B_n , $\hat{V}_n = 0$ on ∂B_n , $\hat{V}_n(0) = 1$, and

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u \hat{V}_n(x) + c(x, u)\hat{V}_n(x)] = \hat{\lambda}_n \hat{V}_n(x) \quad \text{a.e. } x \in B_n.$$

For $v \in \overline{\mathfrak{U}}_{\text{SM}}$ we have

$$\mathcal{L}_v V + c_v V = a^{ij} \partial_{ij} V + b_v \cdot \nabla V + c_v V = \hat{\lambda} V \quad \text{on } \mathbb{R}^d. \quad (4.7)$$

By [Corollary 2.1](#) we obtain $\hat{\lambda} \geq \lambda_v^*(c_v)$. Also by [Theorem 3.2](#) we have $\lambda_v^*(c_v) = \mathcal{E}_x^v(c_v)$ for all $x \in \mathbb{R}^d$. Combining these estimates with [\(4.6\)](#) we obtain

$$\Lambda_x^* \leq \mathcal{E}_x^v(c_v) = \lambda_v^*(c_v) \leq \hat{\lambda} \leq \inf_{z \in \mathbb{R}^d} \Lambda_z^* \quad \forall x \in \mathbb{R}^d.$$

This of course shows that $\hat{\lambda} = \lambda_v^*(c_v) = \Lambda_x^*$ for all $x \in \mathbb{R}^d$, and also proves part (a).

We continue with part (b). By [Theorem 3.2](#) we have

$$\lambda_v^*(c_v - h) < \lambda_v^*(c_v) \quad \forall h \in C_0^+(\mathbb{R}^d), \quad \forall v \in \mathfrak{U}_{\text{SM}}. \quad (4.8)$$

In turn, by [Lemma 2.4](#) there exists a unique eigenfunction $\Psi_v \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ which is associated with the principal eigenvalue $\lambda_v^*(c_v)$ of the operator $\mathcal{L}_v^{c_v} = \mathcal{L}_v + c_v$. Since $\hat{\lambda} = \lambda_v^*(c_v)$ for all $v \in \overline{\mathfrak{U}}_{\text{SM}}$ by part (a), it follows by [\(4.7\)](#) that

$$V = \Psi_v \quad \forall v \in \overline{\mathfrak{U}}_{\text{SM}}. \quad (4.9)$$

By [\(4.8\)](#) and [Lemma 2.2](#) (ii), and since [\(4.3\)](#) is recurrent, we have

$$\Psi_v(x) = \mathbb{E}_x^v \left[e^{\int_0^{\tau} [c_v(X_s) - \Lambda^*] ds} \Psi_v(X_{\tau}) \right] \quad \forall x \in \mathcal{B}^c, \quad \forall v \in \mathfrak{U}_{\text{SM}}^*, \quad (4.10)$$

and all sufficiently large balls \mathcal{B} centered at 0, where $\tau = \tau(\mathcal{B}^c)$, as usual.

Since the Dirichlet eigenvalues satisfy $\hat{\lambda}_n < \hat{\lambda} = \Lambda^*$ for all $n \in \mathbb{N}$, the Dirichlet problem

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u \varphi_n(x) + (c(x, u) - \Lambda^*) \varphi_n(x)] = -\alpha_n \mathbf{1}_{\mathcal{B}}(x) \quad \text{a.e. } x \in B_n, \quad \varphi_n = 0 \quad \text{on } \partial B_n, \quad (4.11)$$

with $\alpha_n > 0$, has a unique solution $\varphi_n \in \mathcal{W}_{\text{loc}}^{2,p}(B_n) \cap C(\bar{B}_n)$, for any $p \geq 1$ [[39](#), Theorem 1.9] (see also [[42](#), Theorem 1.1 (ii)]). We choose α_n as follows: with $\tilde{\alpha}_n > 0$ such that the solution φ_n of [\(4.11\)](#) with $\alpha_n = \tilde{\alpha}_n$ satisfies $\varphi_n(0) = 1$,

we set $\alpha_n = \min(1, \tilde{\alpha}_n)$. Passing to the limit in (4.11) as $n \rightarrow \infty$ along a subsequence, we obtain a nonnegative solution $\Phi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$ of

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u \Phi(x) + (c(x, u) - \Lambda^*) \Phi(x)] = -\alpha \mathbf{1}_{\mathcal{B}}(x), \quad x \in \mathbb{R}^d. \quad (4.12)$$

It is evident from the construction that if $\alpha = 0$ then $\Phi(0) = 1$. On the other hand, if $\alpha > 0$, then necessarily Φ is positive on \mathbb{R}^d . Let $\hat{v} \in \mathcal{U}_{\text{SM}}$ be a selector from the minimizer of (4.12). If $\alpha > 0$, then (4.12) implies that there exists $h \in C_0^+(\mathbb{R}^d)$ such that $\lambda_{\hat{v}}^*(c_{\hat{v}} + h) \leq \Lambda^*$. Since $\lambda_{\hat{v}}^*(c_{\hat{v}}) = \mathcal{E}_{\hat{v}}^*(c_{\hat{v}})$ for all $x \in \mathbb{R}^d$ by Theorem 3.2, and $\mathcal{E}_{\hat{v}}^*(c_{\hat{v}}) \geq \Lambda^*$, then, in view of Corollary 2.1, this contradicts (4.8) and the convexity of $\lambda_{\hat{v}}^*$. Therefore we must have $\alpha = 0$. Let $\bar{v} \in \mathcal{U}_{\text{SM}}^*$. Applying the Itô–Krylov’s formula to (4.11), we obtain

$$\begin{aligned} \varphi_n(x) &\leq \mathbb{E}_x^{\bar{v}} \left[e^{\int_0^{\check{\tau}} [c_{\bar{v}}(X_s) - \Lambda^*] ds} \varphi_n(X_{\check{\tau}}) \mathbf{1}_{\{\check{\tau} < T \wedge \tau_n\}} \right] \\ &\quad + \mathbb{E}_x^{\bar{v}} \left[e^{\int_0^T [c_{\bar{v}}(X_s) - \Lambda^*] ds} \varphi_n(X_T) \mathbf{1}_{\{T < \check{\tau} \wedge \tau_n\}} \right] \quad \forall x \in B_n \setminus \mathcal{B}, \quad \forall T > 0, \end{aligned}$$

where $\check{\tau} = \tau(\mathcal{B}^c)$. Using the argument in the proof of [1, Lemma 2.11], we obtain

$$\Phi(x) \leq \mathbb{E}_x^{\bar{v}} \left[e^{\int_0^{\check{\tau}} [c_{\bar{v}}(X_s) - \Lambda^*] ds} \Phi(X_{\check{\tau}}) \right] \quad \forall x \in \mathcal{B}^c, \quad \forall \bar{v} \in \mathcal{U}_{\text{SM}}^*. \quad (4.13)$$

Comparing (4.10) and (4.13), it follows that, given any $\bar{v} \in \mathcal{U}_{\text{SM}}^*$, we can scale $\Psi_{\bar{v}}$ by a positive constant so that it touches Φ from above at some point in $\bar{\mathcal{B}}$. However, \bar{v} satisfies

$$\mathcal{L}_{\bar{v}} \Phi + c_{\bar{v}} \Phi \geq \Lambda^* \Phi \quad \text{a.e. in } \mathbb{R}^d$$

by (4.12). Thus we have

$$\mathcal{L}_{\bar{v}}(\Psi_{\bar{v}} - \Phi) - (c_{\bar{v}} - \Lambda^*)^-(\Psi_{\bar{v}} - \Phi) \leq 0 \quad \text{a.e. in } \mathbb{R}^d,$$

and it follows by the strong maximum principle that $\Phi = \Psi_{\bar{v}}$ for all $\bar{v} \in \mathcal{U}_{\text{SM}}^*$. Since $\bar{\mathcal{U}}_{\text{SM}} \subset \mathcal{U}_{\text{SM}}^*$ by part (a), it then follows by (4.9) that $V = \Psi_{\bar{v}}$ for all $\bar{v} \in \mathcal{U}_{\text{SM}}^*$. Thus we have

$$\mathcal{L}_{\bar{v}} V + c_{\bar{v}} V = \mathcal{L}_{\bar{v}} \Psi_{\bar{v}} + c_{\bar{v}} \Psi_{\bar{v}} = \lambda_{\bar{v}}^*(c_{\bar{v}}) \Psi_{\bar{v}} = \Lambda^* V = \min_{u \in \mathbb{U}} [\mathcal{L}_u V + c(\cdot, u) V].$$

This proves the verification of optimality result in part (b).

Suppose now that $\tilde{V} \in C^2(\mathbb{R}^d)$ is a positive solution of

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u \tilde{V} + c(\cdot, u) \tilde{V}] = \Lambda^* \tilde{V} \quad \text{on } \mathbb{R}^d. \quad (4.14)$$

Let $\tilde{v} \in \mathcal{U}_{\text{SM}}$ be a selector from the minimizer of (4.14). We have $\lambda_{\tilde{v}}^*(c_{\tilde{v}}) = \mathcal{E}_{\tilde{v}}^*(c_{\tilde{v}}) \geq \Lambda^*$ for all $x \in \mathbb{R}^d$ by Theorem 3.2 and the definition of Λ^* , and $\lambda_{\tilde{v}}^*(c_{\tilde{v}}) \leq \Lambda^*$ by Corollary 2.1. Thus $\mathcal{E}_{\tilde{v}}^*(c_{\tilde{v}}) = \Lambda^*$ for all $x \in \mathbb{R}^d$, which implies that $\tilde{v} \in \mathcal{U}_{\text{SM}}^*$. Then $\tilde{V} = \Psi_{\tilde{v}}$ by the uniqueness of the latter. Therefore, $\tilde{V} = \Psi_{\tilde{v}} = V$ by part (b). This completes the proof. \square

As mentioned in Remark 3.4 the existence of an inf-compact ℓ in Assumption 4.1 is not possible when a, b are bounded. So we consider the following alternate assumption.

Assumption 4.2 There exists a function $\mathcal{V} \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$, such that $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, satisfying

$$\max_{u \in \mathbb{U}} \mathcal{L}_u \mathcal{V}(x) \leq \kappa_0 \mathbf{1}_{\mathcal{K}}(x) - \gamma \mathcal{V}(x), \quad x \in \mathbb{R}^d,$$

for some compact set \mathcal{K} , and positive constants κ_0 and γ , and $c \in \mathfrak{C}$ satisfies $\limsup_{|x| \rightarrow \infty} \sup_{u \in \mathbb{U}} c(x, u) < \gamma$.

A similar condition is used in [10] where the author has obtained only the existence of the function V , and optimal control. Also it is shown in [10] that there exists a constant γ_1 , depending on γ , so that if $\|c\|_{\infty} < \gamma_1$ equation (4.15) below has a solution. We improve these results substantially by proving uniqueness of the solution V , and verification of optimal controls.

Theorem 4.2 Under [Assumption 4.2](#), there exists a positive solution $V \in C^2(\mathbb{R}^d)$ satisfying

$$\min_{u \in \mathbb{U}} [\mathcal{L}_u V + c(\cdot, u)V] = \Lambda^* V. \quad (4.15)$$

Let $\overline{\mathfrak{U}}_{\text{SM}} \subset \mathfrak{U}_{\text{SM}}$ be as in [Theorem 4.1](#). Then (a) and (b) of [Theorem 4.1](#) hold, and (4.15) has a unique positive solution in $C^2(\mathbb{R}^d)$ up to a multiplicative constant.

Proof Part (a) follows exactly as in the proof of [Theorem 4.1](#).

By [Theorem 3.3](#) and [Lemma 2.7](#) for any $v \in \mathfrak{U}_{\text{SM}}$ there exists a unique eigenpair (Ψ_v, λ_v^*) for $\mathcal{L}_v^{c_v}$. In addition,

$$\Psi_v(x) = \mathbb{E}_x^v \left[e^{\int_0^{\tau_r} [c_v(X_s) - \lambda_v^*] ds} \Psi_v(X_{\tau_r}) \right], \quad x \in \bar{B}_r^c.$$

The rest follows as in [Theorem 4.1](#). \square

4.4 Continuity results

It is known from [2] that the set of relaxed stationary Markov controls \mathfrak{U}_{SM} is compactly metrizable (see also [15] for a detailed construction of this topology). In particular $v_n \rightarrow v$ in \mathfrak{U}_{SM} if and only if

$$\int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x, u) v_n(du|x) dx \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x, u) v(du|x) dx$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $g \in C_b(\mathbb{R}^d \times \mathbb{U})$. For $v \in \mathfrak{U}_{\text{SM}}$ we denote by $(\Psi_v, \lambda_v^*(f))$ the principal eigenpair of the operator \mathcal{L}_v^f , i.e.,

$$\mathcal{L}_v \Psi_v(x) + f(x) \Psi_v(x) = \lambda_v^*(f) \Psi_v(x), \quad \Psi_v(x) > 0, \quad x \in \mathbb{R}^d.$$

When $f = c_v$, we occasionally drop the dependence on c_v and denote the eigenvalue as $\lambda_v^* = \lambda_v^*(c_v)$. The next result concerns the continuity of λ_v^* with respect to stationary Markov controls, and extends the result in [8, Proposition 9.2]. The continuity result in [8, Proposition 9.2] is established with respect to the L^∞ norm convergence of the coefficients, whereas [Theorem 4.3](#) that follows is obtained under very weak topological convergence of the coefficients.

Theorem 4.3 Assume one of the following.

- (i) [Assumption 4.1](#) holds, and $c \in \mathcal{C}_{\beta\ell}$ for some $\beta \in (0, 1)$.
- (ii) [Assumption 4.2](#) holds.

Then the map $v \mapsto \lambda_v^*$ is continuous.

Proof We show the proof only under (i). For case (ii) the proof is analogous. Let $v_n \rightarrow v$ in the topology of Markov controls. Let (Ψ_n, λ_n^*) be the principal eigenpair which satisfies

$$\mathcal{L}_{v_n} \Psi_n(x) + c_{v_n}(x) \Psi_n(x) = \lambda_n^* \Psi_n(x), \quad x \in \mathbb{R}^d, \quad \text{and} \quad \lambda_n^* = \mathcal{E}^{v_n}(c_{v_n}), \quad (4.16)$$

where the second display is a consequence of [Theorem 3.2](#) and [Proposition 3.1](#). It is obvious that $\lambda_n^* \geq 0$ for all n .

Since $(\ell(\cdot) - \max_{u \in \mathbb{U}} c(\cdot, u))$ inf-compact, we can find a constant κ_1 such that $\max_{u \in \mathbb{U}} c(x, u) \leq \kappa_1 + \ell(x)$. Since $\mathcal{E}^v(\ell) < \bar{\kappa}_0$ for all $v \in \mathfrak{U}_{\text{SM}}$, we obtain $\lambda_n^* \leq \kappa_1 + \bar{\kappa}_0$ for all n . Thus $\{\lambda_n^* : n \geq 1\}$ is a compact set. Therefore passing to a subsequence we may assume that $\lambda_n^* \rightarrow \lambda^*$ as $n \rightarrow \infty$. To complete the proof we only need to show that $\lambda^* = \lambda_v^*$. Since $\Psi_n(0) = 1$ for all n , and the coefficients b_{v_n} and c_{v_n} are uniformly locally bounded, applying Harnack's inequality and Sobolev's estimate we can find $\Psi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p \geq 1$, such that $\Psi_n \rightarrow \Psi$ weakly in $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$. Therefore by [2, Lemma 2.4.3] and (4.16) we obtain

$$\mathcal{L} \Psi(x) + c_v(x) \Psi(x) = \lambda^* \Psi(x), \quad x \in \mathbb{R}^d, \quad \Psi > 0. \quad (4.17)$$

By [Corollary 2.1](#) we have $\lambda^* \geq \lambda_v^*$.

Let $\mathcal{B} \supset \mathcal{K}$ be an open ball such that $|c(x, u) - \lambda^*| \leq \beta \ell(x)$ for all $(x, u) \in \mathcal{B}^c \times \mathbb{U}$. Let $R > 0$ be large enough so that $\mathcal{B} \subset B_R$ and τ_R denote the exit time from the ball B_R . Applying Itô–Krylov’s formula to [\(4.17\)](#) we obtain

$$\Psi(x) = \mathbb{E}_x^v \left[e^{\int_0^{\tau \wedge \tau_R \wedge T} (c_v(X_s) - \lambda^*) ds} \Psi(X_{\tau \wedge \tau_R \wedge T}) \right], \quad x \in \mathcal{B}^c \cap B_R, \quad (4.18)$$

for any $T > 0$. Since

$$\begin{aligned} \mathbb{E}_x^v \left[e^{\int_0^{\tau} (c_v(X_s) - \lambda^*) ds} \right] &\leq \mathbb{E}_x^v \left[e^{\int_0^{\tau} \beta \ell(X_s) ds} \right] \\ &\leq \left(\mathbb{E}_x^v \left[e^{\int_0^{\tau} \ell(X_s) ds} \right] \right)^\beta < \infty \quad \text{for } x \in \mathcal{B}^c, \end{aligned} \quad (4.19)$$

and Ψ is bounded in $\mathcal{B}^c \cap B_R$, for every fixed R , letting $T \rightarrow \infty$ in [\(4.18\)](#) we have

$$\Psi(x) = \mathbb{E}_x^v \left[e^{\int_0^{\tau \wedge \tau_R} (c_v(X_s) - \lambda^*) ds} \Psi(X_{\tau \wedge \tau_R}) \right], \quad x \in \mathcal{B}^c \cap B_R. \quad (4.20)$$

[Equation \(4.19\)](#) which holds for every $v \in \mathfrak{U}_{\text{SM}}$ and λ_n , possibly for a larger ball \mathcal{B} , shows that, for some constant $\tilde{\kappa}$, we have $\Psi_n(x) \leq \tilde{\kappa}(\mathcal{V}(x))^\beta$ for all $n \in \mathbb{N}$, and $x \in \mathcal{B}^c$. Therefore, $\Psi(x) \leq \tilde{\kappa}(\mathcal{V}(x))^\beta$ for all $x \in \mathcal{B}^c$.

We write

$$\mathbb{E}_x^v \left[e^{\int_0^{\tau \wedge \tau_R} \ell(X_s) ds} \right] = \mathbb{E}_x^v \left[e^{\int_0^{\tau} \ell(X_s) ds} \mathbf{1}_{\{\tau < \tau_R\}} \right] + \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathbf{1}_{\{\tau_R < \tau\}} \right]. \quad (4.21)$$

The left hand side of [\(4.21\)](#) and the first term on the right hand side both converge to $\mathbb{E}_x^v \left[e^{\int_0^{\tau} \ell(X_s) ds} \right]$ as $R \rightarrow \infty$, by monotone convergence. Therefore we have

$$\mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathbf{1}_{\{\tau_R < \tau\}} \right] \xrightarrow{R \rightarrow \infty} 0. \quad (4.22)$$

On the other hand [Assumption 4.1](#) implies that

$$\mathbb{E}_x \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathcal{V}(X_{\tau_R}) \mathbf{1}_{\{\tau_R < \tau\}} \right] \leq \mathcal{V}(x) \quad \forall x \in B_R \setminus \mathcal{B}^c, \quad \forall R > 0$$

We proceed as in the proof of [Theorem 2.5](#). Let $\Gamma(R, m) := \{x \in \partial B_R : \Psi(x) \geq m\}$ for $m \geq 1$. Then

$$\begin{aligned} \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} (c_v(X_s) - \lambda^*) ds} \Psi(X_{\tau_R}) \mathbf{1}_{\{\tau_R < \tau\}} \right] &\leq m \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathbf{1}_{\{\tau_R < \tau\}} \right] \\ &\quad + \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \Psi(X_{\tau_R}) \mathbf{1}_{\Gamma(R, m)}(X_{\tau_R}) \mathbf{1}_{\{\tau_R < \tau\}} \right] \\ &\leq m \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathbf{1}_{\{\tau_R < \tau\}} \right] \\ &\quad + \tilde{\kappa}' m^{\beta-1} \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathcal{V}(X_{\tau_R}) \mathbf{1}_{\{\tau_R < \tau\}} \right] \\ &\leq m \mathbb{E}_x^v \left[e^{\int_0^{\tau_R} \ell(X_s) ds} \mathbf{1}_{\{\tau_R < \tau\}} \right] + \tilde{\kappa} m^{\beta-1} \mathcal{V}(x), \end{aligned} \quad (4.23)$$

for some $\tilde{\kappa}'$, and by first letting $R \rightarrow \infty$, using [\(4.22\)](#), and then $m \rightarrow \infty$, it follows that the left hand side of [\(4.23\)](#) vanishes as $R \rightarrow \infty$. Therefore, letting $R \rightarrow \infty$ in [\(4.20\)](#), we obtain

$$\Psi(x) = \mathbb{E}_x^v \left[e^{\int_0^{\tau} (c_v(X_s) - \lambda^*) ds} \Psi(X_\tau) \right], \quad x \in \mathcal{B}^c.$$

It then follows by [Corollary 2.3](#) that $\lambda^* = \lambda_v^*$, and this completes the proof. \square

Remark 4.1 Following the proof of [Theorem 4.3](#) we can obtain the following continuity result which should be compared with [\[8, Proposition 9.2 \(ii\)\]](#). Consider a sequence of operators $\mathcal{L}_n^{f_n}$ with coefficients (a_n, b_n, f_n) , where b_n, f_n are locally bounded uniformly in n , and $\inf_n(\inf_{\mathbb{R}^d} f_n) > -\infty$. The coefficients a_n and b_n are assumed to satisfy (A1)–(A3) uniformly in n . Assume that $a_n \rightarrow a$ in $C_{\text{loc}}(\mathbb{R}^d)$, and $b_n \rightarrow b$ and $f_n \rightarrow f$ weakly in $L_{\text{loc}}^1(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d} f > -\infty$. Moreover we suppose that one of the following hold

- (a) There exists an inf-compact function $\ell \in C(\mathbb{R}^d)$ and $\mathcal{V} \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$, $\inf_{\mathbb{R}^d} \mathcal{V} > 0$, such that $\mathcal{L}_n \mathcal{V} \leq \bar{\kappa} \mathbf{1}_{\mathcal{K}} - \ell \mathcal{V}$ a.e. on \mathbb{R}^d for some constant $\bar{\kappa}$, and a compact set \mathcal{K} . In addition, $\beta \ell - \sup_n f_n$ is inf-compact for some $\beta \in (0, 1)$.
- (b) The sequence \mathcal{L}_n satisfies (3.14) for all n , $\lim_{n \rightarrow \infty} \|f_n^-\|_{\infty} = \|f^-\|_{\infty}$, and

$$\limsup_{|x| \rightarrow \infty} \sup_n f_n(x) + \|f_n^-\|_{\infty} < \gamma.$$

Then the principal eigenvalue $\lambda^*(f_n)$ converges to $\lambda^*(f)$ as $n \rightarrow \infty$.

As an application of [Theorem 4.3](#) we have the following existence result for the risk-sensitive control problem under (Markovian) risk-sensitive type constraints.

Theorem 4.4 *Assume one of the following.*

- (i) [Assumption 4.1](#) holds, and $c, r_1, \dots, r_m \in \mathcal{C}_{\beta\ell}$ for some $\beta \in (0, 1)$.
- (ii) [Assumption 4.2](#) holds, and $r_1, \dots, r_m \in \mathfrak{C}$ satisfy

$$\max_{i=1, \dots, m} \left\{ \limsup_{|x| \rightarrow \infty} \max_{u \in \mathbb{U}} r_i(x, u) \right\} < \gamma.$$

In addition, suppose that K_i , $i = 1, \dots, m$, are closed subsets of \mathbb{R} , and that there exists $\hat{v} \in \mathfrak{U}_{\text{SM}}$ such that $\mathcal{E}^{\hat{v}}(r_{i,\hat{v}}) \in K_i$ for all i , where we use the usual notation $r_{i,v}(x) := r_i(x, v(x))$.

Then the following constrained minimization problem admits an optimal control in \mathfrak{U}_{SM}

$$\text{minimize over } v \in \mathfrak{U}_{\text{SM}} : \mathcal{E}^v(c_v), \quad \text{subject to } \mathcal{E}^v(r_{i,v}) \in K_i, \quad i = 1, \dots, m.$$

Proof Let $v_n \in \mathfrak{U}_{\text{SM}}$ be a sequence of controls along which the constraints are met, and $\mathcal{E}^{v_n}(c_{v_n})$ converges to its infimum. Since \mathfrak{U}_{SM} is compact under the topology of Markov controls, we may assume, without loss of generality, that v_n converges to some $\bar{v} \in \mathfrak{U}_{\text{SM}}$ as $n \rightarrow \infty$. By [Theorem 4.3](#) we know that $v \mapsto \lambda_v^*(c_v)$, and $v \mapsto \lambda_v^*(r_{i,v})$, $i = 1, \dots, m$, are continuous maps, and that $\mathcal{E}^v(c_v) = \lambda_v^*(c_v)$, and $\mathcal{E}^v(r_{i,v}) = \lambda_v^*(r_{i,v})$ for $i = 1, \dots, m$. It follows that the constraints are met at \bar{v} . Therefore, \bar{v} is an optimal Markov control for the constrained problem. \square

Another application of [Theorem 4.3](#) is a following characterization of λ^* which provides a positive answer to [8, Conjecture 1.8] for a certain class of a, b and f . In [Theorem 4.5](#) below, we consider the uncontrolled generator \mathcal{L} in [section 3](#). Let us introduce the following definition from [8]

$$\lambda'(f) = \sup \left\{ \lambda : \exists \varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d), \varphi > 0, \mathcal{L}\varphi + (f - \lambda)\varphi \geq 0 \text{ a.e. in } \mathbb{R}^d \right\}.$$

Recall the definition of λ'' in (2.46). From [8, Theorem 1.7], under (A1)–(A2), we have $\lambda^*(f) \leq \lambda'(f) \leq \lambda''(f)$ whenever f is bounded above. It is conjectured in [8, Conjecture 1.8] that for bounded a, b , and f , one has $\lambda'(f) = \lambda''(f)$. It should be noted from [Example 3.1](#) that $\lambda^*(f)$ could be strictly smaller than $\lambda''(f)$. The following result complements those in [8, Theorems 1.7 and 1.9].

Theorem 4.5 *For a locally bounded function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with $\inf_{\mathbb{R}^d} f > -\infty$, the following are true.*

- (i) Suppose that $\mathcal{E}_x(f) < \infty$. Then under (A1)–(A3) we have

$$\lambda^*(f) \leq \lambda'(f) \leq \mathcal{E}_x(f) \leq \lambda''(f).$$

- (ii) Let \mathcal{L} , \mathcal{V} and γ satisfy (3.14), and suppose that $\sup_{\mathbb{R}^d} (f + \|f^-\|_{\infty}) < \gamma$. Then $\lambda^*(f) = \lambda''(f)$.

- (iii) Let \mathcal{L} , \mathcal{V} and ℓ satisfy (4.4), and suppose that $\beta \ell - f$ is inf-compact for some $\beta \in (0, 1)$. Then $\lambda^*(f) = \lambda''(f)$.

Proof We first show (i). By [8, Theorem 1.7 (ii)] we have $\lambda^*(f) \leq \lambda'(f)$. Consider $\varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that

$$\mathcal{L}\varphi + (f - \lambda)\varphi \geq 0.$$

Recall that τ_n is the exit time from the open ball $B_n(0)$. Therefore applying Itô-Krylov's formula we obtain

$$\varphi(x) \leq \mathbb{E}_x \left[e^{\int_0^{\tau_n \wedge T} [f(X_s) - \lambda] ds} \varphi(X_{\tau_n \wedge T}) \right] \leq (\sup_{\mathbb{R}^d} \varphi) \mathbb{E}_x \left[e^{\int_0^{\tau_n \wedge T} [f(X_s) - \lambda] ds} \right], \quad T \geq 0. \quad (4.24)$$

Since $\mathcal{E}_x(f)$ is finite, letting $n \rightarrow \infty$ in (4.24), taking logarithms on both sides, dividing by T and then letting $T \rightarrow \infty$ we get $\lambda \leq \mathcal{E}_x(f)$. This implies $\lambda'(f) \leq \mathcal{E}_x(f)$. Now suppose $\varphi \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$, with $\inf_{\mathbb{R}^d} \varphi > 0$, satisfies

$$\mathcal{L}\varphi + (f - \lambda)\varphi \leq 0.$$

Repeating the analogous calculation as above, we obtain $\lambda \geq \mathcal{E}_x(f)$, which implies that $\mathcal{E}_x(f) \leq \lambda''(f)$.

Next we prove (ii). Since $\lambda^*(f + c) = \lambda^*(f) + c$ for any constant c , we may replace f by $f + \|f^-\|_\infty$. Therefore f is non-negative and $\|f\|_\infty < \gamma$. By (i) above we have $\lambda^*(f) \leq \lambda''(f)$. Let $\chi_n: \mathbb{R}^d \rightarrow [0, 1]$ be a cut-off function such that $\chi_n(x) = 1$ for $|x| \leq n$, and $\chi_n(x) = 0$ for $|x| \geq n + 1$. Define $f_n = \chi_n f + (1 - \chi_n)\|f\|_\infty$. Let $(\Psi_n^*, \lambda^*(f_n))$ denote the principal eigenpair of \mathcal{L}^{f_n} . By Remark 4.1 we have $\lambda^*(f_n) \rightarrow \lambda^*(f)$ as $n \rightarrow \infty$. Thus to complete the proof it is enough to show that $\inf_{\mathbb{R}^d} \Psi_n^* > 0$, which implies that $\lambda^*(f_n) = \lambda''(f_n) \geq \lambda''(f)$ for all n , and thus $\lambda^*(f) \geq \lambda''(f)$. Note that $\lambda^*(f_n) \leq \mathcal{E}(f_n) \leq \|f\|_\infty$ for all n . Now fix n and let τ_n be the first hitting time to the ball B_n . Then applying the Itô-Krylov formula to

$$\mathcal{L}\Psi_n^* + (f_n - \lambda^*(f_n))\Psi_n^* = 0,$$

together with Fatou's lemma we have

$$\min_{z \in B_{n+1}} \Psi_n^*(z) \leq \mathbb{E}_x \left[e^{\int_0^{\tau_n} [f_n(X_s) - \lambda^*(f_n)] ds} \Psi_n^*(X_{\tau_n}) \right] \leq \Psi_n^*(x)$$

for all $x \in \overline{B}_{n+1}^c(0)$. Hence $\inf_{\mathbb{R}^d} \Psi_n^* > 0$ which completes the proof.

The proof of (iii) is completely analogous to the proof of part (ii). Since $\beta\ell - f^+$ is inf-compact, we can find $g: \mathbb{R}^d \rightarrow \mathbb{R}_+$, such that $\lim_{|x| \rightarrow \infty} g(x) = \infty$, and $\beta\ell - f^+ - g$ is inf-compact. We let $f_n = \chi_n f + (1 - \chi_n)(g + f^+)$. Note that $\inf_n (\beta\ell - f_n) = \inf_{n \in \mathbb{N}} (\chi_n(\beta\ell - f) + (1 - \chi_n)(\beta\ell - f^+ - g))$ is inf-compact. On the other hand, $f_n \geq f$ for all n . The rest follows as part (ii). \square

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